

Zero-electron-mass limit of Euler-Poisson equations

Jiang Xu^{1*}, Ting Zhang^{2†}

1. Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, P.R.China

2. Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Abstract

This paper is concerned with multidimensional Euler-Poisson equations for plasmas. The equations take the form of Euler equations for the conservation laws of the mass density and current density for charge-carriers (electrons and ions), coupled to a Poisson equation for the electrostatic potential. We study the limit to zero of some physical parameters which arise in the scaled Euler-Poisson equations, more precisely, which is the limit of vanishing ratio of the electron mass to the ion mass. When the initial data are small in critical Besov spaces, by virtue of the “Shizuta-Kawashima” skew-symmetry condition, we establish the uniform global existence and uniqueness of classical solutions. Then we develop new frequency-localization Strichartz-type estimates for the equation of acoustics (a modified wave equation) with the aid of the detailed analysis of the semigroup formulation generated by this modified wave operator. Finally, it is shown that the uniform classical solutions converge towards that of the incompressible Euler equations (for *ill-prepared* initial data) in a refined way as the scaled electron-mass tends to zero.

Keyword: Zero-electron-mass limits, skew-symmetry, Euler-Poisson equations, critical Besov spaces, Strichartz-type estimate

1 Introduction

The Euler-Poisson equations, sometimes called as the (macroscopic) hydrodynamic models, are to describe the transport of charge-carriers (electrons and ions) in semiconductor devices or plasmas. The system consists of the conservation laws for the mass density and current density for carriers, with a Poisson equation for the electrostatic potential. Using the classical moment method, it is derived from the semi-classical Boltzmann-Poisson equations, for details, see [14]. In recent three decades, the Euler-Poisson equations has attracted increasing attention both numerical simulation and theoretical analysis, since it can capture more physical phenomena that occur in submicron devices or in the presence of high electric fields than the traditional drift-diffusion models.

More specifically, we consider an un-magnetized plasmas consisting of electrons with charge $q_e = -1$ and of a single species of ions with charge $q_i = +1$. Denote by $n_e = n_e(t, x)$, $\mathbf{v}_e = \mathbf{v}_e(t, x)$ (resp., n_i, \mathbf{v}_i) the scaled density and mean velocity of the electrons (resp., ions) and by $\phi = \phi(t, x)$ the scaled electrostatic potential. From [9], these unknowns satisfy the following scaled Euler-Poisson equations:

$$\begin{cases} \frac{\partial}{\partial t} n_a + \operatorname{div}(n_a \mathbf{v}_a) = 0, \\ \delta_a \frac{\partial}{\partial t} (n_a \mathbf{v}_a) + \delta_a \operatorname{div}(n_a \mathbf{v}_a \otimes \mathbf{v}_a) + \nabla P_a(n_a) = -q_a n_a \nabla \phi - \delta_a \frac{n_a \mathbf{v}_a}{\tau_a}, \\ \lambda^2 \Delta \phi = n_e - n_i + C(x), \end{cases} \quad (1.1)$$

where $a = e, i$ and $(t, x) \in [0, +\infty) \times \mathbb{R}^N (N \geq 2)$. The symbols $\operatorname{div}, \nabla, \Delta$ and \otimes are the x -divergence operator, gradient operator, Laplacian operator and the tensor product of two vectors respectively; the parameters $\lambda, \tau_a > 0$ are the scaled constants for the Debye length and the momentum relaxation time

*jiangxu_79@yahoo.com.cn

†zhangting79@zju.edu.cn

of electrons (if $a = e$; otherwise, ions if $a = i$) respectively; the pressure $P_a(n_a)$ is a smooth function satisfying

$$P'_a(n_a) > 0 \quad \text{for all } n_a > 0.$$

For the sake of (mathematical) simplicity, we assume that it satisfies the usual “ γ -law”

$$P_a(n_a) = c_a n_a^{\gamma_a},$$

where $c_a > 0$ is a physical constant. In such case, the plasmas is called *isothermal* if $\gamma_a = 1$ and *isentropic* if $\gamma_a > 1$. The function $C(x)$, which only depends on the space variable, represents the density of fixed charged background ions (doping profile).

The dimensionless parameter δ_a in the Euler-Poisson equations (1.1) is given by

$$\delta_a = \frac{m_a v_0^2}{k_B T_0},$$

where m_e (resp. m_i) is the mass of a single electron (resp. ion), k_B is the Boltzmann constant, v_0 and T_0 are typical velocity and temperature values for the plasmas respectively. The reader is refer to [9] for details about the scaling and the physical assumptions. Since m_i is not smaller than the mass of a proton, we have

$$\frac{m_e}{m_i} \leq \frac{m_e}{\text{the mass of a proton}} \simeq 5.45 \times 10^{-4}.$$

Namely, $\frac{m_e}{m_i}$ is a small number. Thus, if v_0^2 is chosen to be $k_B T_0 / m_i$, then $\delta_i = 1$ and $\delta_e = \frac{m_e}{m_i} \ll 1$. The main goal of this paper is to investigate the limit as δ_e goes to zero.

In plasmas physics, the zero-electron-mass assumption (*i.e.* $\delta_e = 0$) is widely used, *e.g.*, see [7, 10]. For simplicity, we assume that $\bar{n} := n_i - C(x)$ is a given positive constant and therefore consider the unipolar Euler-Poisson equations only, instead of (1.1):

$$\begin{cases} n_t + \operatorname{div}(n\mathbf{v}) = 0 \\ (n\mathbf{v})_t + \operatorname{div}(n\mathbf{v} \otimes \mathbf{v}) + \frac{\nabla P(n)}{\epsilon^2} = \frac{n\nabla\phi}{\epsilon^2} - n\mathbf{v} \\ \Delta\phi = n - \bar{n}, \end{cases} \quad (1.2)$$

with the initial data

$$(n, \mathbf{v})(x, 0) = (n_0, \mathbf{v}_0), \quad (1.3)$$

where we set $\tau_e = 1 = \lambda$ and

$$n = n_e, \quad \mathbf{v} = \mathbf{v}_e, \quad \delta_e = \epsilon^2, \quad P = P_e = An^\gamma.$$

At the formal level, if $\nabla P(n) \rightarrow 0, \nabla\phi \rightarrow 0$, when ϵ goes to zero. Hence n must be the positive constant \bar{n} . Then passing to the limit in the mass conservation equation, we get $\operatorname{div}\mathbf{v} \rightarrow 0$. Coming back to the momentum equation, we conclude that \mathbf{u} (the limit of \mathbf{v}) must satisfy the incompressible Euler equations with damping

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} + \nabla \Pi = 0, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = u_0, \end{cases}$$

which also can be written

$$\begin{cases} \mathbf{u}_t + \mathcal{P}(\mathbf{u} \cdot \nabla \mathbf{u}) + \mathbf{u} = 0, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = u_0 = \mathcal{P}\mathbf{v}_0, \end{cases} \quad (1.4)$$

where \mathcal{P} stands for the Leray projector on solenoidal vector fields.

Moreover, if we assume that $n = \bar{n} + O(\epsilon^2), \operatorname{div}\mathbf{v} = O(\epsilon)$, then $\nabla P(n) = O(\epsilon^2), \nabla\phi = O(\epsilon^2)$. Indeed, this entails that n_t, \mathbf{v}_t are uniformly bounded so that the dangerous time oscillations can not occur. Starting from this simple consideration, Ali, Chen, Jüngel and Peng [1] justified the zero-electron-mass limit ($\epsilon \rightarrow 0$) of (1.2) with data of the following type

$$n_0 = \bar{n} + \epsilon^2 n_{0,1}, \quad \mathbf{v}_0 = \bar{\mathbf{v}}(x, 0) + \epsilon \mathbf{v}_{0,1}$$

with $\operatorname{div} \bar{\mathbf{v}}(x, 0) = 0$, and $(n_{0,1}, \mathbf{v}_{0,1})$ uniformly bounded in the periodic Sobolev space $H^\sigma(\mathbb{T}^N)$ ($\sigma > 1 + N/2$). Such data are referred as *well-prepared* data in the usual PDE's terminology. In the case of *well-prepared* data, the zero-electron-mass limit has some similarities with the classical low-Mach-number limit in the compressible Euler equations studied by Klainerman and Majda [11, 12]. The unique difference is that there is an extra singularity from the electron-field term. Ali *et al.* [1] overcame this difficulty by a careful use of the mass conservation and the Poisson equation, and obtained the uniform *a priori* estimates with respect to the scaled zero-electron-mass. Then they justified the zero-electron-mass limit for smooth solutions from (1.2)-(1.3) to (1.4) (incompressible limit) by virtue of the Aubin-Lions compactness lemma. The precise limit behaviors of smooth solutions were shown by the recent work [17]. The zero-electron-mass limit for weak entropy solutions was studied by Goudon, Jüngel and Peng [6] under some restrictive assumptions, with the help of the kinetic formulation, the monotonicity and compactness arguments. It is worth noting that these results only provided *local* convergence or the case of *well-prepared* data.

In the present paper, we shall study the zero-electron-mass limit since few works can be found in mathematics literatures, except [1, 6]. We focus on the case of *ill-prepared* data and hope to get the *global* convergence of classical solutions. We make a weaker assumption that the initial density only satisfies $n_0 = \bar{n} + \epsilon n_{0,1}$ with $(n_{0,1}, \mathbf{v}_0)$ uniformly bounded (in an appropriate functional space).

1.1 Symmetry

It is convenient to state the basic ideas and main results of this paper, we first introduce a function transform to reduce (1.2) to a symmetric hyperbolic-elliptic form.

For the isentropic case ($\gamma > 1$), by defining the sound speed

$$\psi(n) = \sqrt{p'(n)},$$

and the sound speed $\bar{\psi} = \psi(\bar{n})$ at a background density \bar{n} , we set

$$m = \frac{2}{\gamma - 1} \left(\psi(n) - \bar{\psi} \right). \quad (1.5)$$

Then (1.2) can be rewritten as

$$\begin{cases} m_t + \bar{\psi} \operatorname{div} \mathbf{v} = -\mathbf{v} \cdot \nabla m - \frac{\gamma-1}{2} m \operatorname{div} \mathbf{v}, \\ \mathbf{v}_t + \bar{\psi} \epsilon^{-2} \nabla m = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{\gamma-1}{2} \epsilon^{-2} m \nabla m + \epsilon^{-2} \nabla \phi - \mathbf{v}, \\ \Delta \phi = h(m), \end{cases} \quad (1.6)$$

where

$$h(m) = \left\{ (A\gamma)^{-\frac{1}{2}} \left(\frac{\gamma-1}{2} m + \bar{\psi} \right) \right\}^{\frac{2}{\gamma-1}} - \bar{n}$$

is a smooth function on the domain $\{m | \frac{\gamma-1}{2} m + \bar{\psi} > 0\}$ satisfying $h(0) = 0$. The corresponding initial data become

$$(m_0, \mathbf{v}_0, \nabla \phi_0) = \left\{ \frac{2}{\gamma-1} \left(\psi(n_0) - \bar{\psi} \right), \mathbf{v}_0, \nabla \Delta^{-1}(n_0 - \bar{n}) \right\}. \quad (1.7)$$

For the isothermal case where $\gamma = 1$, the form (1.6) is still valid with $\bar{\psi} = \sqrt{A}$. However, it depends on the following enthalpy variable change

$$m = \sqrt{A}(\ln n - \ln \bar{n}), \quad (1.8)$$

which the details are referred to [5]. About the equivalence for classical solutions away from the vacuum between (1.2)-(1.3) and (1.6)-(1.7), see Section 3. Moreover we introduce the variables as follows:

$$m^\epsilon = \frac{m}{\epsilon}, \quad \mathbf{v}^\epsilon = \mathbf{v}, \quad \nabla \phi^\epsilon = \frac{\nabla \phi}{\epsilon}.$$

Then the new variables satisfy

$$\begin{cases} m_t^\epsilon + \bar{\psi}\epsilon^{-1}\operatorname{div}\mathbf{v}^\epsilon = -\mathbf{v}^\epsilon \cdot \nabla m^\epsilon - \frac{\gamma-1}{2}m^\epsilon\operatorname{div}\mathbf{v}^\epsilon, \\ \mathbf{v}_t^\epsilon + \bar{\psi}\epsilon^{-1}\nabla m^\epsilon = -\mathbf{v}^\epsilon \cdot \nabla \mathbf{v}^\epsilon - \frac{\gamma-1}{2}m^\epsilon\nabla m^\epsilon + \epsilon^{-1}\nabla\phi^\epsilon - \mathbf{v}^\epsilon, \\ \Delta\phi^\epsilon = \epsilon^{-1}h(\epsilon m^\epsilon), \end{cases} \quad (1.9)$$

with the initial data

$$(m_0^\epsilon, \mathbf{v}_0^\epsilon, \nabla\phi_0^\epsilon) = \left\{ \frac{2}{\gamma-1} \left(\frac{\psi(n_0) - \bar{\psi}}{\epsilon} \right), \mathbf{v}_0, \nabla\Delta^{-1} \left(\frac{n_0 - \bar{n}}{\epsilon} \right) \right\}. \quad (1.10)$$

One expects \mathbf{v}^ϵ to tend to \mathbf{u} in (1.9) which \mathbf{u} solves the incompressible Euler equations (1.4) as $\epsilon \rightarrow 0$. The expected convergence however is not easy to justify rigorously. Here, there are two main difficulties. The first is the singularity from the electron-field term, which can not be overcome by using the symmetrizer of main part of hyperbolic system as in [11, 12]. We adopt a new (but small!) technique to deal with the singular electron-field term, which can help us to simplify the similar analysis as [1] heavily, see (4.6)-(4.7). The second is that one has to face the propagation of acoustic waves with the speed ϵ^{-1} , a phenomenon which does not occur in the case of *well-prepared* data. To solve this, we need to develop some new ideas. Inspired by [4], we split the velocity into a divergence-free part and a gradient part to obtain the equation of acoustics

$$\begin{cases} m_t^\epsilon + \bar{\psi}\frac{\Delta d^\epsilon}{\epsilon} = F, \\ d_t^\epsilon + d^\epsilon - \bar{\psi}\frac{\Delta m^\epsilon}{\epsilon} - \frac{h'(0)\Delta^{-1}m^\epsilon}{\epsilon} = G, \end{cases}$$

i.e.,

$$d_{tt}^\epsilon + d_t^\epsilon - \frac{\bar{\psi}^2}{\epsilon^2}\Delta d^\epsilon + \frac{h'(0)\bar{\psi}}{\epsilon^2}d^\epsilon = G_t + \frac{\bar{\psi}\Lambda}{\epsilon}F + \frac{h'(0)\Lambda^{-1}}{\epsilon}F,$$

(for details, see Section 5). Then we establish some dispersive estimates according to the semigroup theory of this modified wave operator, and achieve a new frequency-localization Strichartz-type estimate which is used to pass to the zero-electron-mass limit in a refined way.

Based on the recent work [5, 8], we still choose the critical Besov space framework in space-variable x (a subalgebra of $\mathcal{W}^{1,\infty}$) to study the global well-posedness and the zero-electron-mass limit of classical solutions to the system (1.2)-(1.3). The main results are states as follows.

1.2 Main results

Theorem 1.1. *Set $\sigma = 1 + N/2$. There is a positive constant δ_0 independent of ϵ , such that if*

$$\left\| \left(\frac{n_0 - \bar{n}}{\epsilon}, \mathbf{v}_0, \frac{\mathbf{e}_0}{\epsilon} \right) \right\|_{B_{2,1}^\sigma(\mathbb{R}^N)} \leq \delta_0$$

for $0 < \epsilon \leq \epsilon_0$, and $\mathbf{e}_0 := \nabla\Delta^{-1}(n_0 - \bar{n})$, then the system (1.2)-(1.3) admits a unique global solution $(n, \mathbf{v}, \nabla\phi)$ satisfying

$$(n - \bar{n}, \mathbf{v}, \nabla\phi) \in \mathcal{C}(\mathbb{R}^+, B_{2,1}^\sigma(\mathbb{R}^N)).$$

Moreover, the uniform energy estimate holds:

$$\begin{aligned} & \left\| \left(\frac{n - \bar{n}}{\epsilon}, \mathbf{v}, \frac{\nabla\phi}{\epsilon} \right) (\cdot, t) \right\|_{B_{2,1}^\sigma(\mathbb{R}^N)} \\ & \leq C_0 \left\| \left(\frac{n_0 - \bar{n}}{\epsilon}, \mathbf{v}_0, \frac{\mathbf{e}_0}{\epsilon} \right) \right\|_{B_{2,1}^\sigma(\mathbb{R}^N)} \exp(-\mu_0 t), \quad t \geq 0, \end{aligned}$$

where μ_0, C_0 are some positive constants independent of ϵ .

Remark 1.1. The symbol $\nabla\Delta^{-1}$ means

$$\nabla\Delta^{-1}f = \int_{\mathbb{R}^d} \nabla_x G(x-y)f(y)dy,$$

where $G(x, y)$ is a solution to $\Delta_x G(x, y) = \delta(x-y)$ with $x, y \in \mathbb{R}^N$. In the periodic setting, the regularity assumption on \mathbf{e}_0 can be removed.

Remark 1.2. In the proof of Theorem 1.1, different from that in [5], “Shizuta-Kawashima” skew-symmetry condition developed for general hyperbolic systems of balance laws [13, 19] is used, which helps us avoid differentiating the system with respect to time-variable t and the proof is shortened. Thanks to the isentropic Euler-Poisson equations also including isentropic Euler equations [3], the concrete information of skew-symmetry matrix $K(\xi)$ is well known (unknown for general systems) which is very effective to estimate the coupled electron-field $\nabla\phi^\epsilon$, see (4.21).

Theorem 1.2. *Let the assumptions of Theorem 1.1 be fulfilled. Let $(m^\epsilon, \mathbf{v}^\epsilon, \nabla\phi^\epsilon)$ denote the global solution to (1.9)-(1.10), then $(m^\epsilon, \mathcal{Q}\mathbf{v}^\epsilon, \nabla\phi^\epsilon)$ tends to zero in $L^1(\mathbb{R}^+; B_{p,1}^{N/p}(\mathbb{R}^N))$ and $\mathcal{P}\mathbf{v}^\epsilon$ converges in $L^\infty(\mathbb{R}^+; B_{p,1}^{N/p}(\mathbb{R}^N)) \cap L^1(\mathbb{R}^+; B_{p,1}^{N/p}(\mathbb{R}^N))$ towards the solution $\mathbf{u} \in (L^\infty \cap L^1)(\mathbb{R}^+; B_{2,1}^\sigma(\mathbb{R}^N))$ to the incompressible Euler equations (1.4), as $\epsilon \rightarrow 0$. Here $2 \leq p \leq \infty$, \mathcal{P} stands for the Leray projector on solenoidal vector fields and is defined by $\mathcal{P} = I - \mathcal{Q}$ with $\mathcal{Q} = \nabla\Delta^{-1}\text{div}$.*

Remark 1.3. The general convergence statements are give by Proposition 5.4 and Proposition 5.6, where the speed of convergence may be characterized in terms of power of ϵ .

Remark 1.4. Let us mention that the approach developed by this paper can also be applied to study the low-Mach-number limits of the compressible Euler equations with damping for the perfect gas flow. Therefore, this work can be regarded as a supplement to the theory of asymptotic limits for hyperbolic problems.

Remark 1.5. Prescribing *ill-prepared* initial data and periodic boundary conditions preclude from using dispersive properties in order to pass to the limit in (1.9). Indeed, there is no chance that the acoustic waves go at infinity. Hence, there could be resonances which may hinder the convergence to the incompressible Euler equations. This will be shown in a forthcoming paper.

Our paper is organized as follows. In Section 2, we introduce Besov spaces and their properties. In Section 3, we give some remarks on the hyperbolic symmetrization and recall a local existence result of classical solutions. In Section 4, we derive the *a-priori* estimate which is used to achieve the global existence of uniform classical solutions. For clarity, Section 5 is divided into two parts. Based on the detailed analysis of the equation of acoustics, we first establish a new frequency-localization Strichartz-type estimate. Then using this estimate, we perform the zero-electron-mass limit in a refined way.

Finally, some efforts on other kinds of asymptotic limits (such as relaxation-time limit and quasineutral limit) of classical solutions to the Euler-Poisson equations (1.2)-(1.3) should be mentioned, the interested reader is referred to [15, 16, 18] and the literature quoted therein.

2 Preliminary

Throughout this paper, C is a uniform positive constant independent of ϵ . $f \approx g$ means that $f \leq Cg$ and $g \leq Cf$. We denote by $L^\rho(0, T; X)$, $\mathcal{C}([0, T], X)$ (resp., $\mathcal{C}^1([0, T], X)$) the space of ρ -power integrable and continuation (resp., continuously differentiable) functions on $[0, T]$ with values in a Banach space X , respectively. In the case $T = +\infty$, we sometimes label $L^\rho(\mathbb{R}^+; X)$, $\mathcal{C}(\mathbb{R}^+, X)$, etc. For brevity, we use the notation $\|(a, b, c)\|_X := \|a\|_X + \|b\|_X + \|c\|_X$, where $a, b, c \in X$. All functional spaces of the present paper are considered in \mathbb{R}^N , so we may omit the space dependence for simplicity.

In this section, we review briefly the Littlewood–Paley decomposition theory and the characterization of Besov spaces; see also, *e.g.*, [4] or [5].

Let \mathcal{S} be the Schwarz class. (φ, χ) is a couple of smooth functions valued in $[0, 1]$ such that φ is supported in the shell $\mathbf{C}(0, \frac{3}{4}, \frac{8}{3}) = \{\xi \in \mathbb{R}^N | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, χ is supported in the ball $\mathbf{B}(0, \frac{4}{3}) = \{\xi \in \mathbb{R}^N | |\xi| \leq \frac{4}{3}\}$ satisfying

$$\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad q \in \mathbb{N}, \quad \xi \in \mathbb{R}^N$$

and

$$\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \quad k \in \mathbb{Z}, \quad \xi \in \mathbb{R}^N \setminus \{0\}.$$

For $f \in \mathcal{S}'$ (denote the set of temperate distributions which is the dual of \mathcal{S}), one can define the Fourier dyadic blocks as follows:

$$\begin{aligned}\Delta_{-1}f &:= \chi(D)f = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}f), \quad \Delta_q f := 0 \quad \text{for } q \leq -2; \\ \Delta_q f &:= \varphi(2^{-q}D)f = \mathcal{F}^{-1}(\varphi(2^{-q}|\xi|)\mathcal{F}f) \quad \text{for } q \geq 0; \\ \dot{\Delta}_k f &:= \varphi(2^{-k}D)f = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\mathcal{F}f) \quad \text{for } k \in \mathbb{Z},\end{aligned}$$

where $\mathcal{F}f$, $\mathcal{F}^{-1}f$ represent the Fourier transform and the inverse Fourier transform on f , respectively. The nonhomogeneous Littlewood–Paley decomposition is

$$f = \sum_{q \geq -1} \Delta_q f \quad \text{in } \mathcal{S}'.$$

Define the low-frequency cut-off by

$$S_q f := \sum_{p \leq q-1} \Delta_p f.$$

The above Littlewood–Paley decomposition is almost orthogonal in L^2 .

Proposition 2.1. *For any $f, g \in \mathcal{S}'$, the following properties hold:*

$$\Delta_p \Delta_q f \equiv 0 \quad \text{if } |p - q| \geq 2,$$

$$\Delta_q (S_{p-1} f \Delta_p g) \equiv 0 \quad \text{if } |p - q| \geq 5.$$

Based on the above Littlewood–Paley decomposition, we introduce the explicit definitions of nonhomogeneous Besov spaces.

Definition 2.1. Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For $1 \leq r < \infty$, the Besov spaces $B_{p,r}^s$ are defined by

$$f \in B_{p,r}^s \Leftrightarrow \left(\sum_{q \geq -1} (2^{qs} \|\Delta_q f\|_{L^p})^r \right)^{\frac{1}{r}} < \infty$$

and $B_{p,\infty}^s$ are defined by

$$f \in B_{p,\infty}^s \Leftrightarrow \sup_{q \geq -1} 2^{qs} \|\Delta_q f\|_{L^p} < \infty.$$

Some conclusions as follows will be used in subsequent analysis. The first one is the classical Bernstein's inequality.

Lemma 2.2 (Bernstein's inequality). *Let $k \in \mathbb{N}$ and $0 < R_1 < R_2$. There exists a constant C depending only on R_1, R_2 , and N such that for all $1 \leq a \leq b \leq \infty$ and $f \in L^a$, we have*

$$\text{Supp } \mathcal{F}f \subset B(0, R_1 \lambda) \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+N(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a},$$

$$\text{Supp } \mathcal{F}f \subset C(0, R_1 \lambda, R_2 \lambda) \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^{k+1} \lambda^k \|f\|_{L^a},$$

where $\mathcal{F}f$ (or $\hat{f} = \int_{\mathbb{R}^N} f(x) \exp(-ix \cdot \xi) dx$) represents the Fourier transform on f .

The second one is the embedding properties in Besov spaces.

Lemma 2.3.

$$B_{p,r}^s \hookrightarrow B_{p,\tilde{r}}^{\tilde{s}} \quad \text{whenever } \tilde{s} < s \text{ or } \tilde{s} = s \text{ and } r \leq \tilde{r};$$

$$B_{p,r}^s \hookrightarrow B_{\tilde{p},r}^{s-N(\frac{1}{p}-\frac{1}{\tilde{p}})} \quad \text{whenever } \tilde{p} > p;$$

$$B_{p,1}^{d/p} (1 \leq p < \infty) \hookrightarrow \mathcal{C}_0, \quad B_{\infty,1}^0 \hookrightarrow \mathcal{C} \cap L^\infty,$$

where \mathcal{C}_0 is the space of continuous bounded functions which decay at infinity.

The three one is the compactness result for Besov spaces.

Proposition 2.4. *Let $1 \leq p, r \leq \infty$, $s \in \mathbb{R}$, and $\epsilon > 0$. For all $\phi \in C_c^\infty$, the map $f \mapsto \phi f$ is compact from $B_{p,r}^{s+\epsilon}$ to $B_{p,r}^s$.*

The last one is a continuity result for compositions.

Proposition 2.5. *Let $1 \leq p, r \leq \infty$, and I be an open interval of \mathbb{R} . Let $s > 0$ and let n be the smallest integer such that $n \geq s$. Let $F : I \rightarrow \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in W^{n,\infty}(I; \mathbb{R})$. Assume that $v \in B_{p,r}^s$ takes values in $J \subset \subset I$. Then $F(v) \in B_{p,r}^s$ and there exists a constant C depending only on s, I, J , and N such that*

$$\|F(v)\|_{B_{p,r}^s} \leq C(1 + \|v\|_{L^\infty})^n \|F'\|_{W^{n,\infty}(I)} \|v\|_{B_{p,r}^s}.$$

3 Local existence

In order to obtain the effective *a priori* estimates by the low- and high-frequency decomposition methods, we formulate (1.2) into the symmetric hyperbolic-elliptic form (1.6) in the Introduction.

Here, we give some remarks.

Remark 3.1. ($\gamma > 1$) (1.5) induces a variable change from the half-space $\{(n, \mathbf{v}, \nabla \phi) \in (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N\}$ to the open set $\{(m, \mathbf{v}, \nabla \phi) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \mid \frac{\gamma-1}{2}m + \bar{\psi} > 0\}$. It is easy to show that for classical solutions $(n, \mathbf{v}, \nabla \phi)$ away from the vacuum, the system (1.2) is equivalent to (1.6).

For the isothermal case ($\gamma = 1$), we also have the similar equivalence. Note that $\sqrt{A} \ln n$ is the enthalpy, instead of the sound speed.

Remark 3.2. (1.8) induces a variable change from the half-space $\{(n, \mathbf{v}, \nabla \phi) \in (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N\}$ to the whole space $\{(m, \mathbf{v}, \nabla \phi) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N\}$. It is easy to show that for classical solutions $(n, \mathbf{v}, \nabla \phi)$ away from the vacuum, the system (1.2) is equivalent to (1.6).

The symmetric hyperbolic system (1.9) can also be rewrite as the vector form

$$\partial_t W^\epsilon + \sum_{j=1}^N A_j^\epsilon(\mathbf{v}^\epsilon) \partial_{x_j} W^\epsilon = \begin{pmatrix} -\frac{\gamma-1}{2} m^\epsilon \operatorname{div} \mathbf{v}^\epsilon \\ -\mathbf{v}^\epsilon - \frac{\gamma-1}{2} m^\epsilon \nabla m^\epsilon + \epsilon^{-1} \nabla \phi^\epsilon \end{pmatrix}, \quad (3.1)$$

coupled with the dynamic electron-potential equation

$$\Delta \phi^\epsilon = \epsilon^{-1} h(\epsilon m^\epsilon), \quad (3.2)$$

where

$$W^\epsilon = \begin{pmatrix} m^\epsilon \\ \mathbf{v}^\epsilon \end{pmatrix}, \quad A_j^\epsilon(\mathbf{v}^\epsilon) = \begin{pmatrix} v^{\epsilon j} & \frac{\bar{\psi}}{\epsilon} e_j^\top \\ \frac{\bar{\psi}}{\epsilon} e_j & v^{\epsilon j} I_{N \times N} \end{pmatrix}$$

($I_{N \times N}$ denotes the unit matrix of order N)

and e_j is N -dimensional vector where the j th component is one, others are zero).

Now, we recall a local existence result on classical solutions to (1.9)-(1.10), which has been obtained in [5].

Proposition 3.1. *For any fixed $\epsilon \in (0, 1]$, suppose that $(m_0^\epsilon, \mathbf{v}_0^\epsilon, \nabla \phi_0^\epsilon) \in B_{2,1}^\sigma$ satisfying $\frac{\gamma-1}{2} \epsilon m_0^\epsilon + \bar{\psi} > 0$, then there exist a time $T_0 > 0$ and a unique solution $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)$ to (1.9)-(1.10) such that $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon) \in \mathcal{C}^1([0, T_0] \times \mathbb{R}^N)$ with $\frac{\gamma-1}{2} \epsilon m^\epsilon + \bar{\psi} > 0$ for all $t \in [0, T_0]$ and $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon) \in \mathcal{C}([0, T_0], B_{2,1}^\sigma) \cap \mathcal{C}^1([0, T_0], B_{2,1}^{\sigma-1})$.*

4 A uniform *a priori* estimate

In this section, we establish a crucial *a priori* estimate by using the low- and high-frequency decomposition methods, which is used to derive the global existence and exponential stability of classical solutions to (1.9)-(1.10).

Proposition 4.1. *There exist three positive constants δ_1, C_1 and μ_1 independent of ϵ , such that for any $T > 0$, if*

$$\sup_{0 \leq t \leq T} \|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)(\cdot, t)\|_{B_{2,1}^\sigma} \leq \delta_1, \quad (4.1)$$

then

$$\|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)(\cdot, t)\|_{B_{2,1}^\sigma} \leq C_1 \left\| \left(\frac{m_0}{\epsilon}, \mathbf{v}_0, \frac{\mathbf{e}_0}{\epsilon} \right) \right\|_{B_{2,1}^\sigma} \exp(-\mu_1 t), \quad (4.2)$$

where $t \in [0, T]$.

Having this Proposition 4.1, we can extend the local-in-time solutions in Proposition 3.1 by virtue of the standard continuation argument and obtain the global existence of uniform classical solutions to the system (1.9)-(1.10). Using the imbedding property in Besov space $B_{2,1}^\sigma$, we know $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon) \in \mathcal{C}^1([0, \infty) \times \mathbb{R}^N)$ solves (1.9)-(1.10). The choice of δ_1 is sufficient to ensure $\frac{\gamma-1}{2}\epsilon m^\epsilon + \bar{\psi} > 0$ for $0 < \epsilon \leq 1$. From the Remark 3.1, we achieve that $(n, \mathbf{v}, \nabla \phi) \in \mathcal{C}^1([0, \infty) \times \mathbb{R}^N)$ is a solution of (1.2)-(1.3) with $n > 0$. Furthermore, we arrive at Theorem 1.1.

The main ingredients in the proof of Proposition 4.1 are the high-frequency ($q \geq 0$) estimates and low-frequency ($q = -1$) estimates on $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)$. We divide it into several lemmas, since the proof is a bit longer.

Lemma 4.2. *If $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon) \in \mathcal{C}([0, T], B_{2,1}^\sigma) \cap \mathcal{C}^1([0, T], B_{2,1}^{\sigma-1})$ is a solution of (1.9)-(1.10) for any given $T > 0$, then the following estimate holds ($q \geq -1$):*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{n} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \right) + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 \\ & \leq C \left(\|\Delta_q m^\epsilon, \Delta_q \mathbf{v}^\epsilon, \Delta_q \nabla \phi^\epsilon\|_{L^2} \left\{ \|(\nabla m^\epsilon, \nabla \mathbf{v}^\epsilon)\|_{L^\infty} \|(\Delta_q m^\epsilon, \Delta_q \mathbf{v}^\epsilon)\|_{L^2} \right. \right. \\ & \quad + \|[\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla m^\epsilon\|_{L^2} + \|[\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla \mathbf{v}^\epsilon\|_{L^2} + \|[m^\epsilon, \Delta_q] \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} \\ & \quad \left. \left. + \|[m^\epsilon, \Delta_q] \nabla m^\epsilon\|_{L^2} + \frac{1}{\epsilon} \|\Delta_q (h(\epsilon m^\epsilon) \mathbf{v}^\epsilon)\|_{L^2} \right\} \right), \end{aligned} \quad (4.3)$$

where the commutator $[f, g] := fg - gf$ and C is a uniform positive constant independent of ϵ .

Proof. Applying the localization operator Δ_q to (1.9) gives

$$\begin{cases} \partial_t \Delta_q m^\epsilon + \bar{\psi} \epsilon^{-1} \operatorname{div} \Delta_q \mathbf{v}^\epsilon + (\mathbf{v}^\epsilon \cdot \nabla) \Delta_q m^\epsilon \\ = [\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla m^\epsilon - \frac{\gamma-1}{2} m^\epsilon \Delta_q \operatorname{div} \mathbf{v}^\epsilon + \frac{\gamma-1}{2} [m^\epsilon, \Delta_q] \operatorname{div} \mathbf{v}^\epsilon, \\ \\ \partial_t \Delta_q \mathbf{v}^\epsilon + \bar{\psi} \epsilon^{-1} \nabla \Delta_q m^\epsilon + (\mathbf{v}^\epsilon \cdot \nabla) \Delta_q \mathbf{v}^\epsilon \\ = [\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla \mathbf{v}^\epsilon - \frac{\gamma-1}{2} m^\epsilon \Delta_q \nabla m^\epsilon + \frac{\gamma-1}{2} [m^\epsilon, \Delta_q] \nabla m^\epsilon \\ \quad + \epsilon^{-1} \nabla \Delta_q \phi^\epsilon - \Delta_q \mathbf{v}^\epsilon \\ \\ \Delta_q \Delta \phi^\epsilon = \epsilon^{-1} \Delta_q h(\epsilon m^\epsilon). \end{cases} \quad (4.4)$$

Then, by multiplying the first equation of (4.4) by $\Delta_q m^\epsilon$, the second one by $\Delta_q \mathbf{v}^\epsilon$ respectively, and adding the two resulting equations together, then integrating the resulting equations over \mathbb{R}^N , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 \right) + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 \\ & = \int \operatorname{div} \mathbf{v}^\epsilon (|\Delta_q m^\epsilon|^2 + |\Delta_q \mathbf{v}^\epsilon|^2) \end{aligned}$$

$$\begin{aligned}
& + \int ([\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla m^\epsilon \Delta_q m^\epsilon + [\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla \mathbf{v}^\epsilon \Delta_q \mathbf{v}^\epsilon) \\
& + \frac{\gamma-1}{2} \int \Delta_q m^\epsilon (\nabla m^\epsilon \cdot \Delta_q \mathbf{v}^\epsilon) + \frac{\gamma-1}{2} \int [m^\epsilon, \Delta_q] \operatorname{div} \mathbf{v}^\epsilon \cdot \Delta_q m^\epsilon \\
& + \frac{\gamma-1}{2} \int [m^\epsilon, \Delta_q] \nabla m^\epsilon \cdot \Delta_q \mathbf{v}^\epsilon + \frac{1}{\epsilon} \int \nabla \Delta_q \phi^\epsilon \cdot \Delta_q \mathbf{v}^\epsilon.
\end{aligned} \tag{4.5}$$

In above equality (4.5), we may use the spectral localization mass equation and Poisson equation in (4.4) in order to eliminate the singularity from the electron-field term similar to the idea in [1], but this will cause very tedious calculations. Here, we observe an equality

$$\operatorname{div} \mathbf{v}^\epsilon = - \frac{\epsilon \operatorname{div} \nabla \phi_t^\epsilon + \operatorname{div}(h(\epsilon m^\epsilon) \mathbf{v}^\epsilon)}{\bar{n}} \tag{4.6}$$

following from the first equation and the third one in (1.2) under the symmetrization. Hence, we deduce that

$$\begin{aligned}
& \frac{1}{\epsilon} \int \nabla \Delta_q \phi^\epsilon \cdot \Delta_q \mathbf{v}^\epsilon \\
& = - \frac{1}{\epsilon} \int \Delta_q \phi^\epsilon \Delta_q \operatorname{div} \mathbf{v}^\epsilon \\
& = \frac{1}{\bar{n}\epsilon} \int \Delta_q \phi^\epsilon \Delta_q \left(\epsilon \operatorname{div} \nabla \phi_t^\epsilon + \operatorname{div}(h(\epsilon m^\epsilon) \mathbf{v}^\epsilon) \right) \\
& = - \frac{1}{2\bar{n}} \frac{d}{dt} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 - \frac{1}{\bar{n}\epsilon} \int \Delta_q \nabla \phi^\epsilon \Delta_q (h(\epsilon m^\epsilon) \mathbf{v}^\epsilon), \\
& \leq - \frac{1}{2\bar{n}} \frac{d}{dt} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 + \frac{1}{\bar{n}\epsilon} \|\Delta_q (h(\epsilon m^\epsilon) \mathbf{v}^\epsilon)\|_{L^2} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}.
\end{aligned} \tag{4.7}$$

Together with (4.5) and (4.7), we arrive at (4.3) immediately with the aid of Cauchy-Schwartz inequality. \square

In this position, we formulate an important skew-symmetry lemma which has been well developed in [3, 13, 19], which is sometimes referred to as the ‘‘Kawashima condition’’.

Lemma 4.3 (Shizuta-Kawashima). *For all $\xi \in \mathbb{R}^N$, $\xi \neq 0$, there exists a real skew-symmetric smooth matrix $K(\xi)$ which is defined in the unit sphere \mathcal{S}^{N-1} :*

$$K(\xi) = \begin{pmatrix} 0 & \frac{\xi^\top}{|\xi|} \\ -\frac{\xi}{|\xi|} & 0 \end{pmatrix}, \tag{4.8}$$

such that

$$K(\xi) \sum_{j=1}^N \xi_j A_j^\epsilon(0) = \begin{pmatrix} \frac{\bar{\psi}}{\epsilon} |\xi| & 0 \\ 0 & -\frac{\bar{\psi}}{\epsilon} \frac{\xi \otimes \xi}{|\xi|} \end{pmatrix}, \tag{4.9}$$

where A_j^ϵ is the matrix appearing in the system (3.1).

Due to the skew-symmetry structure of the system (3.1), we can develop some new frequency-localization estimates and avoid performing the t -derivative to (3.1) as in [5].

Lemma 4.4. *If $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon) \in \mathcal{C}([0, T], B_{2,1}^\sigma) \cap \mathcal{C}^1([0, T], B_{2,1}^{\sigma-1})$ is a solution of (1.9)-(1.10) for any given $T > 0$, then the following estimates hold:*

$$\begin{aligned}
& \frac{\epsilon}{2} \frac{d}{dt} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi + \frac{\bar{\psi}}{2} 2^{2q} \|\Delta_q m^\epsilon\|_{L^2}^2 \\
& \leq C 2^{2q} \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + C \epsilon 2^q \|\Delta_q W^\epsilon\|_{L^2} (\|\Delta_q \mathcal{G}\|_{L^2} + \|m^\epsilon\|_{L^\infty} \|\Delta_q \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} \\
& \quad + \|[m^\epsilon, \Delta_q] \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} + \|m^\epsilon\|_{L^\infty} \|\Delta_q \nabla m^\epsilon\|_{L^2} + \|[m^\epsilon, \Delta_q] \nabla m^\epsilon\|_{L^2})
\end{aligned}$$

$$+C\|\Delta_q(H(\epsilon m^\epsilon)m^\epsilon)\|_{L^2}\|\Delta_q m^\epsilon\|_{L^2} \quad (q \geq 0); \quad (4.10)$$

$$\begin{aligned} & \frac{\epsilon}{2} \frac{d}{dt} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_{-1} W^\epsilon})^* K(\xi) \widehat{\Delta_{-1} W^\epsilon} \right) d\xi + (A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} \|\Delta_{-1} m^\epsilon\|_{L^2}^2 \\ \leq & C\|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2}^2 + C\epsilon\|\Delta_{-1} W^\epsilon\|_{L^2} (\|\Delta_{-1} \mathcal{G}\|_{L^2} + \|m^\epsilon\|_{L^\infty} \|\Delta_{-1} \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} \\ & + \|[m^\epsilon, \Delta_{-1}] \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} + \|m^\epsilon\|_{L^\infty} \|\Delta_{-1} \nabla m^\epsilon\|_{L^2} + \|[m^\epsilon, \Delta_{-1}] \nabla m^\epsilon\|_{L^2}) \\ & + C\|\Delta_{-1}(H(\epsilon m^\epsilon)m^\epsilon)\|_{L^2}\|\Delta_{-1} m^\epsilon\|_{L^2}. \end{aligned} \quad (4.11)$$

where the function \mathcal{G} is given by (4.13), $H(m) = \int_0^1 h'(\varsigma m) d\varsigma - (A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}}$ is a smooth function on $\{m | \varsigma m + \bar{h} > 0, \varsigma \in [0, 1]\}$ satisfying $H(0) = 0$ and C is a uniform positive constant independent of ϵ .

Proof. The system (3.1) can be written as the linearized form

$$\partial_t W^\epsilon + \sum_{j=1}^N A_j^\epsilon(0) \partial_{x_j} W^\epsilon = \mathcal{G} + \begin{pmatrix} -\frac{\gamma-1}{2} m^\epsilon \operatorname{div} \mathbf{v}^\epsilon \\ -\mathbf{v}^\epsilon - \frac{\gamma-1}{2} m^\epsilon \nabla m^\epsilon + \frac{1}{\epsilon} \nabla \phi^\epsilon \end{pmatrix}, \quad (4.12)$$

where

$$\mathcal{G} = \sum_{j=1}^N \left\{ A_j^\epsilon(0) - A_j^\epsilon(\mathbf{v}^\epsilon) \right\} \partial_{x_j} W^\epsilon. \quad (4.13)$$

Applying the operator Δ_q to the system (4.12) gives

$$\begin{aligned} & \partial_t \Delta_q W^\epsilon + \sum_{j=1}^N A_j^\epsilon(0) \partial_{x_j} \Delta_q W^\epsilon \\ = & \Delta_q \mathcal{G} + \begin{pmatrix} -\frac{\gamma-1}{2} \Delta_q(m^\epsilon \operatorname{div} \mathbf{v}^\epsilon) \\ -\Delta_q \mathbf{v}^\epsilon - \frac{\gamma-1}{2} \Delta_q(m^\epsilon \nabla m^\epsilon) + \frac{1}{\epsilon} \Delta_q \nabla \phi^\epsilon \end{pmatrix}, \end{aligned} \quad (4.14)$$

By performing the Fourier transform with respect to the space variable x for (4.14) and multiplying the resulting equation by $-i\epsilon(\widehat{\Delta_q W^\epsilon})^* K(\xi)$ ($*$ represents transpose and conjugate), then taking the real part of each term in the equality, we can obtain

$$\begin{aligned} & \epsilon \operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \frac{d}{dt} \widehat{\Delta_q W^\epsilon} \right) + \epsilon (\widehat{\Delta_q W^\epsilon})^* K(\xi) \left(\sum_{j=1}^N \xi_j A_j^\epsilon(0) \right) \widehat{\Delta_q W^\epsilon} \\ = & \epsilon \operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) (\widehat{\Delta_q \mathcal{G}}) \right) - \epsilon \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \mathbf{v}^\epsilon} \right) \\ & + \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \nabla \phi^\epsilon} \right) + \frac{(\gamma-1)\epsilon}{2} \operatorname{Im} \left(\widehat{\Delta_q \mathbf{v}^\epsilon} \cdot \frac{\xi}{|\xi|} (\Delta_q(\widehat{m^\epsilon \operatorname{div} \mathbf{v}^\epsilon})) \right) \\ & - \frac{(\gamma-1)\epsilon}{2} \operatorname{Im} \left(\widehat{\Delta_q m^\epsilon} \frac{\xi^\top}{|\xi|} (\Delta_q(\widehat{m^\epsilon \nabla m^\epsilon})) \right). \end{aligned} \quad (4.15)$$

Using the skew-symmetry of $K(\xi)$, we have

$$\operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \frac{d}{dt} \widehat{\Delta_q W^\epsilon} \right) = \frac{1}{2} \frac{d}{dt} \operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right). \quad (4.16)$$

Substituting (4.9) into the second term on the left-hand side of (4.15), it is not difficult to get

$$\begin{aligned} & \epsilon \operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \frac{d}{dt} \widehat{\Delta_q W^\epsilon} \right) + \epsilon (\widehat{\Delta_q W^\epsilon})^* K(\xi) \left(\sum_{j=1}^N \xi_j A_j^\epsilon(0) \right) \widehat{\Delta_q W^\epsilon} \\ \geq & \frac{\epsilon}{2} \frac{d}{dt} \operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) + \bar{\psi} |\xi| |\widehat{\Delta_q W^\epsilon}|^2 - 2\bar{\psi} |\xi| |\widehat{\Delta_q \mathbf{v}^\epsilon}|^2. \end{aligned} \quad (4.17)$$

With the help of Young inequality, the right-hand side of (4.15) can be estimated as

$$\begin{aligned}
& \epsilon \operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) (\widehat{\Delta_q \mathcal{G}}) \right) - \epsilon \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \mathbf{v}^\epsilon} \right) \\
& + \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \nabla \phi^\epsilon} \right) + \frac{(\gamma-1)\epsilon}{2} \operatorname{Im} \left(\widehat{\Delta_q \mathbf{v}^\epsilon} \cdot \frac{\xi}{|\xi|} (\Delta_q (\widehat{m^\epsilon \operatorname{div} \mathbf{v}^\epsilon})) \right) \\
& - \frac{(\gamma-1)\epsilon}{2} \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \frac{\xi^\top}{|\xi|} (\Delta_q (\widehat{m^\epsilon \nabla m^\epsilon})) \right) \\
\leq & \frac{\bar{\psi}}{2} |\xi| |\widehat{\Delta_q W^\epsilon}|^2 + \frac{C}{|\xi|} |\widehat{\Delta_q \mathbf{v}^\epsilon}|^2 + \epsilon |\widehat{\Delta_q W^\epsilon}| |\widehat{\Delta_q \mathcal{G}}| + C \epsilon |\widehat{\Delta_q \mathbf{v}^\epsilon}| |(\Delta_q (\widehat{m^\epsilon \operatorname{div} \mathbf{v}^\epsilon}))| \\
& + C \epsilon |\widehat{\Delta_q m^\epsilon}| |(\Delta_q (\widehat{m^\epsilon \nabla m^\epsilon}))| + \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \nabla \phi^\epsilon} \right), \tag{4.18}
\end{aligned}$$

where we have used the uniform boundedness of the matrix $K(\xi)$ ($\xi \neq 0$). Combining the equality (4.15) and the inequality (4.17)-(4.18), we deduce

$$\begin{aligned}
& \frac{\epsilon}{2} \frac{d}{dt} \operatorname{Im} \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) + \frac{\bar{\psi}}{2} |\xi| |\widehat{\Delta_q W^\epsilon}|^2 \\
\leq & C \left(|\xi| + \frac{1}{|\xi|} \right) |\widehat{\Delta_q \mathbf{v}^\epsilon}|^2 + \epsilon |\widehat{\Delta_q W^\epsilon}| |\widehat{\Delta_q \mathcal{G}}| + C \epsilon |\widehat{\Delta_q \mathbf{v}^\epsilon}| |(\Delta_q (\widehat{m^\epsilon \operatorname{div} \mathbf{v}^\epsilon}))| \\
& + C \epsilon |\widehat{\Delta_q m^\epsilon}| |(\Delta_q (\widehat{m^\epsilon \nabla m^\epsilon}))| + \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \nabla \phi^\epsilon} \right). \tag{4.19}
\end{aligned}$$

Multiplying (4.19) by $|\xi|$ and integrating it over \mathbb{R}^N , using Plancherel's theorem, we obtain

$$\begin{aligned}
& \frac{\epsilon}{2} \frac{d}{dt} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi + \frac{\bar{\psi}}{2} \|\Delta_q \nabla W^\epsilon\|_{L^2}^2 \\
\leq & C(2^{2q} + 1) \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + C \epsilon 2^q \|\Delta_q W^\epsilon\|_{L^2} \|\Delta_q \mathcal{G}\|_{L^2} \\
& + C \epsilon 2^q \|\Delta_q \mathbf{v}^\epsilon\|_{L^2} \|\Delta_q (m^\epsilon \operatorname{div} \mathbf{v}^\epsilon)\|_{L^2} + C \epsilon 2^q \|\Delta_q m^\epsilon\|_{L^2} \|\Delta_q (m^\epsilon \nabla m^\epsilon)\|_{L^2} \\
& + \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \xi^\top \widehat{\Delta_q \nabla \phi^\epsilon} \right) d\xi. \tag{4.20}
\end{aligned}$$

Furthermore, the last term on the right-hand side of (4.20) can be estimated as

$$\begin{aligned}
& \operatorname{Im} \left((\widehat{\Delta_q m^\epsilon}) \xi^\top \widehat{\Delta_q \nabla \phi^\epsilon} \right) d\xi \\
= & -\frac{i}{2} \int \left((\widehat{\Delta_q m^\epsilon}) \xi^\top \widehat{\Delta_q \nabla \phi^\epsilon} \right) d\xi + \frac{i}{2} \int \left((\widehat{\Delta_q m^\epsilon}) \xi^\top \overline{\widehat{\Delta_q \nabla \phi^\epsilon}} \right) d\xi \\
= & \frac{1}{2} \int (\widehat{\Delta_q \nabla m^\epsilon}) \cdot \widehat{\Delta_q \nabla \phi^\epsilon} d\xi + \frac{1}{2} \int (\widehat{\Delta_q \nabla m^\epsilon}) \cdot \overline{\widehat{\Delta_q \nabla \phi^\epsilon}} d\xi \\
= & \frac{(2\pi)^N}{2} \left\{ \int \overline{\Delta_q \nabla m^\epsilon} \cdot \Delta_q \nabla \phi^\epsilon dx + \int \Delta_q \nabla m^\epsilon \cdot \overline{\Delta_q \nabla \phi^\epsilon} dx \right\} \\
= & -\frac{(2\pi)^N}{2} \left\{ \int \overline{\Delta_q m^\epsilon} \Delta_q \Delta \phi^\epsilon dx + \int \Delta_q m^\epsilon \overline{\Delta_q \Delta \phi^\epsilon} dx \right\} \\
= & -\frac{(2\pi)^N}{2\epsilon} \left\{ \int \overline{\Delta_q m^\epsilon} \Delta_q (h(\epsilon m^\epsilon) - h(0)) dx + \int \Delta_q m^\epsilon \overline{\Delta_q (h(\epsilon m^\epsilon) - h(0))} dx \right\} \\
= & -(A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} (2\pi)^N \|\Delta_q m^\epsilon\|_{L^2}^2 \\
& - \frac{(2\pi)^N}{2} \left\{ \int \overline{\Delta_q m^\epsilon} \Delta_q (H(\epsilon m^\epsilon) m^\epsilon) dx + \int \Delta_q m^\epsilon \overline{\Delta_q (H(\epsilon m^\epsilon) m^\epsilon)} dx \right\}, \tag{4.21}
\end{aligned}$$

where $H(m) = \int_0^1 h'(\varsigma m) d\varsigma - (A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}}$ is a smooth function on $\{m | \frac{\gamma-1}{2} \varsigma m + \bar{\psi} > 0, \varsigma \in [0, 1]\}$ satisfying $H(0) = 0$. Therefore, from (4.20)-(4.21), we have

$$\frac{\epsilon}{2} \frac{d}{dt} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi + \frac{\bar{\psi}}{2} \|\Delta_q \nabla W^\epsilon\|_{L^2}^2$$

$$\begin{aligned}
& + (A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} (2\pi)^N \|\Delta_q m^\epsilon\|_{L^2}^2 \\
\leq & C(2^{2q} + 1) \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + C\epsilon 2^q \|\Delta_q W^\epsilon\|_{L^2} \|\Delta_q \mathcal{G}\|_{L^2} \\
& + C\epsilon 2^q \|\Delta_q \mathbf{v}^\epsilon\|_{L^2} \|\Delta_q (m^\epsilon \operatorname{div} \mathbf{v}^\epsilon)\|_{L^2} + C\epsilon 2^q \|\Delta_q m^\epsilon\|_{L^2} \|\Delta_q (m^\epsilon \nabla m^\epsilon)\|_{L^2} \\
& + C \|\Delta_q (H(\epsilon m^\epsilon) m^\epsilon)\|_{L^2} \|\Delta_q m^\epsilon\|_{L^2}.
\end{aligned} \tag{4.22}$$

In view of Lemma 2.2

$$\|\Delta_q \nabla f\|_{L^2} \approx 2^q \|\Delta_q f\|_{L^2} \quad (q \geq 0),$$

we get the estimate (4.10) and (4.11) immediately. \square

On the electron field $\nabla \phi$, we have the following *a priori* estimates.

Lemma 4.5. *If $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon) \in \mathcal{C}([0, T], B_{2,1}^\sigma) \cap \mathcal{C}^1([0, T], B_{2,1}^{\sigma-1})$ is a solution of (1.9)-(1.10) for any given $T > 0$, then*

$$\begin{aligned}
& 2^{2q} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \\
\leq & C \left((A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} \|\Delta_q m^\epsilon\|_{L^2} + \|\Delta_q (H(\epsilon m^\epsilon) m^\epsilon)\|_{L^2} \right) 2^q \|\Delta_q \nabla \phi^\epsilon\|_{L^2} \quad (q \geq 0);
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
& -\epsilon \frac{d}{dt} \int \Delta_{-1} \nabla \phi^\epsilon \cdot \overline{\Delta_{-1} \mathbf{v}^\epsilon} + (A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} \bar{\psi} \|\Delta_{-1} m^\epsilon\|_{L^2}^2 + \|\Delta_{-1} \nabla \phi^\epsilon\|_{L^2}^2 \\
\leq & C(\bar{n} \|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2} + \|\Delta_{-1} (h(\epsilon m^\epsilon) \mathbf{v}^\epsilon)\|_{L^2}) \|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2} \\
& + C \left(\|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2} + \|\mathbf{v}^\epsilon\|_{L^\infty} \|\Delta_{-1} \nabla \mathbf{v}^\epsilon\|_{L^2} + \|[\mathbf{v}^\epsilon, \Delta_{-1}] \nabla \mathbf{v}^\epsilon\|_{L^2} \right. \\
& \left. + \|m^\epsilon\|_{L^\infty} \|\Delta_{-1} \nabla m^\epsilon\|_{L^2} + \|[m^\epsilon, \Delta_{-1}] \nabla m^\epsilon\|_{L^2} \right) \|\Delta_{-1} \nabla \phi^\epsilon\|_{L^2} \\
& + C \|\Delta_{-1} (H(\epsilon m^\epsilon) m^\epsilon)\|_{L^2} \|\Delta_{-1} m^\epsilon\|_{L^2},
\end{aligned} \tag{4.24}$$

where C is a uniform positive constant independent of ϵ .

Proof. By applying the localization operator Δ_q ($q \geq 0$) to both sides of $\operatorname{div} \nabla \phi^\epsilon = \epsilon^{-1} h(\epsilon m^\epsilon)$, integrating it over \mathbb{R}^N after multiplying $\Delta_q \operatorname{div} \nabla \phi^\epsilon$, and noticing the irrotationality of $\nabla \phi^\epsilon$, we can obtain (4.23) in virtue of Hölder's inequality.

From (1.2) and (1.5), we get

$$\nabla \phi_t^\epsilon = -\frac{1}{\epsilon} \nabla \Delta^{-1} \nabla \cdot \{h(\epsilon m^\epsilon) \mathbf{v}^\epsilon + \bar{n} \mathbf{v}^\epsilon\}, \tag{4.25}$$

where the non-local term $\nabla \Delta^{-1} \nabla \cdot f$ is the product of Riesz transforms on f . From (1.9) and (4.25), we have

$$\begin{aligned}
& -\epsilon \frac{d}{dt} \int \Delta_{-1} \nabla \phi^\epsilon \cdot \overline{\Delta_{-1} \mathbf{v}^\epsilon} \\
= & -\epsilon \int \Delta_{-1} \nabla \phi_t^\epsilon \cdot \overline{\Delta_{-1} \mathbf{v}^\epsilon} - \epsilon \int \Delta_{-1} \nabla \phi^\epsilon \cdot \overline{\Delta_{-1} \mathbf{v}_t^\epsilon} \\
= & \mathcal{I} + \int \nabla \Delta^{-1} \nabla \cdot \Delta_{-1} \{h(\epsilon m^\epsilon) \mathbf{v}^\epsilon + \bar{n} \mathbf{v}^\epsilon\} \overline{\Delta_{-1} \mathbf{v}^\epsilon} \\
& -\epsilon \int \Delta_{-1} \nabla \phi^\epsilon \cdot \left(-\overline{\Delta_{-1} \mathbf{v}^\epsilon} - \mathbf{v}^\epsilon \overline{\Delta_{-1} \nabla \mathbf{v}^\epsilon} + \overline{[\mathbf{v}^\epsilon, \Delta_{-1}] \nabla \mathbf{v}^\epsilon} \right. \\
& \left. - \frac{\gamma-1}{2} m \overline{\Delta_{-1} \nabla m^\epsilon} + \frac{\gamma-1}{2} \overline{[m^\epsilon, \Delta_{-1}] \nabla m^\epsilon} + \frac{1}{\epsilon} \overline{\Delta_{-1} \nabla \phi^\epsilon} \right)
\end{aligned} \tag{4.26}$$

where \mathcal{I} can be estimated as

$$\begin{aligned}
\mathcal{I} &= \bar{\psi} \int \Delta_{-1} \nabla \phi^\epsilon \overline{\Delta_{-1} \nabla m^\epsilon} \\
&= -\bar{\psi} \int \Delta_{-1} \Delta \phi^\epsilon \overline{\Delta_{-1} m^\epsilon}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\bar{\psi}}{\epsilon} \int \Delta_{-1} h(\epsilon m^\epsilon) \overline{\Delta_{-1} m^\epsilon} \\
&= -(A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} \bar{\psi} \|\Delta_{-1} m^\epsilon\|_{L^2}^2 - \bar{\psi} \int \Delta_{-1} (H(\epsilon m^\epsilon) m^\epsilon) \overline{\Delta_{-1} m^\epsilon}.
\end{aligned} \tag{4.27}$$

Then using the L^2 -boundedness of Riesz transform and Hölder's inequality, we derive (4.24) immediately. \square

For the estimates of the commutators in (4.3) and (4.10)-(4.11) and (4.24), we have the following conclusion.

Lemma 4.6 (see [5]). *Let $s > 0$ and $1 < p < \infty$; then the following inequalities are true:*

$$\begin{aligned}
&2^{qs} \|[f, \Delta_q] \mathcal{A}g\|_{L^p} \\
&\leq \begin{cases} Cc_q \|f\|_{B_{p,1}^s} \|g\|_{B_{p,1}^s}, & f, g \in B_{p,1}^s, \quad s = 1 + N/p, \\ Cc_q \|f\|_{B_{p,1}^s} \|g\|_{B_{p,1}^{s+1}}, & f \in B_{p,1}^s, \quad g \in B_{p,1}^{s+1}, \quad s = N/p, \\ Cc_q \|f\|_{B_{p,1}^{s+1}} \|g\|_{B_{p,1}^s}, & f \in B_{p,1}^{s+1}, \quad g \in B_{p,1}^s, \quad s = N/p. \end{cases}
\end{aligned}$$

In particular, if $f = g$, then

$$2^{qs} \|[f, \Delta_q] \mathcal{A}g\|_{L^p} \leq Cc_q \|\nabla f\|_{L^\infty} \|g\|_{B_{p,1}^s}, \quad s > 0,$$

where the operator $\mathcal{A} = \text{div}$ or ∇ , C is a harmless constant, and c_q denotes a sequence such that $\|(c_q)\|_{l^1} \leq 1$.

With these lemmas for ready, now, we complete the proof of the uniform *a priori* estimate (4.2).

Proof of Proposition 4.1. Note that the *a priori* assumption (4.1), we deduce from the embedding inequality in Besov spaces that

$$\sup_{0 \leq t \leq T} (\|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)(\cdot, t)\|_{W^{1,\infty}}) \leq C\delta_1. \tag{4.28}$$

To ensure the smoothness of functions $h(\epsilon m^\epsilon)$ and $H(\epsilon m^\epsilon)$, together with the smallness of ϵ , it suffices to choose $0 < \delta_1 \leq \frac{\bar{\psi}}{(\gamma-1)C}$ such that

$$\frac{\gamma-1}{2} \epsilon m^\epsilon(t, x) + \bar{\psi} \geq \frac{\bar{\psi}}{2} > 0, \quad (t, x) \in [0, T] \times \mathbb{R}^N$$

and

$$\frac{\gamma-1}{2} \varsigma \epsilon m^\epsilon(t, x) + \bar{\psi} \geq \frac{\bar{\psi}}{2} > 0, \quad \varsigma \in [0, 1], \quad (t, x) \in [0, T] \times \mathbb{R}^N.$$

For the proof of Proposition 4.1, it can be divided into the following high-frequency part and low-frequency part.

Lemma 4.7 ($q \geq 0$). *There exist some positive constants K_1, K_2, μ_2 independent of ϵ such that the following estimate holds:*

$$\begin{aligned}
&2^{q(\sigma-1)} \frac{d}{dt} \left\{ \frac{K_1}{2} 2^{2q} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{\bar{n}} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \right) \right. \\
&\quad \left. + \frac{K_2 \epsilon}{2} \text{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi \right\}^{1/2} \\
&\quad + \mu_2 2^{q\sigma} \left(\|\Delta_q m^\epsilon\|_{L^2} + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2} + \|\Delta_q \nabla \phi^\epsilon\|_{L^2} \right) \\
&\leq C 2^{q\sigma} \|(W^\epsilon, \nabla W^\epsilon)\|_{L^\infty} \|\Delta_q W^\epsilon\|_{L^2} + Cc_q \|W^\epsilon\|_{B_{2,1}^\sigma}^2 + \frac{C 2^{q\sigma}}{\epsilon} \|\Delta_q (h(\epsilon m^\epsilon) \mathbf{v}^\epsilon)\|_{L^2} \\
&\quad + C 2^{q(\sigma-1)} \|\Delta_q \mathcal{G}\|_{L^2} + C 2^{q\sigma} \|\Delta_q (H(\epsilon m^\epsilon) m^\epsilon)\|_{L^2},
\end{aligned} \tag{4.29}$$

where K_1, K_2 are given by (4.31) and C is a uniform positive constant independent of ϵ .

Proof. Combining (4.3), (4.10) and (4.23), we have

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{K_1}{2} 2^{2q} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{\bar{n}} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \right) \right. \\
& \left. + \frac{K_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi \right\} \\
& + K_1 2^{2q} \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{K_2 \bar{\psi}}{2} 2^{2q} \|\Delta_q m^\epsilon\|_{L^2}^2 + K_3 2^{2q} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \\
\leq & CK_1 2^{2q} \|(\Delta_q m^\epsilon, \Delta_q \mathbf{v}^\epsilon, \Delta_q \nabla \phi^\epsilon)\|_{L^2} \left\{ \|(\nabla m^\epsilon, \nabla \mathbf{v}^\epsilon)\|_{L^\infty} \|(\Delta_q m^\epsilon, \Delta_q \mathbf{v}^\epsilon)\|_{L^2} \right. \\
& + \|[\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla m^\epsilon\|_{L^2} + \|[\mathbf{v}^\epsilon, \Delta_q] \cdot \nabla \mathbf{v}^\epsilon\|_{L^2} + \|[m^\epsilon, \Delta_q] \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} \\
& + \|[m^\epsilon, \Delta_q] \nabla m^\epsilon\|_{L^2} + \frac{1}{\epsilon} \|\Delta_q (h(\epsilon m^\epsilon) \mathbf{v}^\epsilon)\|_{L^2} \left. \right\} + CK_2 2^{2q} \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 \\
& + CK_2 \epsilon^{2q} \|\Delta_q W^\epsilon\|_{L^2} (\|\Delta_q \mathcal{G}\|_{L^2} + \|m^\epsilon\|_{L^\infty} \|\Delta_q \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} \\
& + \|[m^\epsilon, \Delta_q] \operatorname{div} \mathbf{v}^\epsilon\|_{L^2} + \|m^\epsilon\|_{L^\infty} \|\Delta_q \nabla m^\epsilon\|_{L^2} + \|[m^\epsilon, \Delta_q] \nabla m^\epsilon\|_{L^2}) \\
& + CK_2 \|\Delta_q (H(\epsilon m^\epsilon) m^\epsilon)\|_{L^2} \|\Delta_q m^\epsilon\|_{L^2} \\
& + CK_3 \left((A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} \|\Delta_q m^\epsilon\|_{L^2} + \|\Delta_q (H(\epsilon m^\epsilon) m^\epsilon)\|_{L^2} \right) 2^q \|\Delta_q \nabla \phi^\epsilon\|_{L^2},
\end{aligned} \tag{4.30}$$

where the uniform positive constants K_1, K_2 and K_3 (independent of ϵ) satisfy

$$K_2 = \frac{K_1}{4C}, \quad K_3 = \frac{A\gamma \bar{\psi}}{4C^2 \bar{n}^{3-\gamma}} K_2. \tag{4.31}$$

Due to

$$\begin{aligned}
& \left| \frac{K_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi \right| \\
\leq & \frac{CK_2}{2} 2^{2q} (\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2) \quad (0 < \epsilon \leq 1),
\end{aligned} \tag{4.32}$$

so we introduce these uniform constants in order to ensure

$$\begin{aligned}
& \frac{K_1}{2} 2^{2q} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{\bar{n}} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \right) \\
& + \frac{K_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi \\
\approx & 2^{2q} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \right)
\end{aligned} \tag{4.33}$$

and eliminate the quadratic terms

$$CK_3 (A\gamma)^{-\frac{1}{2}} \bar{n}^{\frac{3-\gamma}{2}} 2^q \|\Delta_q m^\epsilon\|_{L^2} \|\Delta_q \nabla \phi^\epsilon\|_{L^2} \quad \text{and} \quad CK_2 2^{2q} \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2$$

in the right-hand side of (4.30) with the aid of Young's inequality, for similar details, see [5]. Dividing the resulting inequality by

$$\begin{aligned}
& \left\{ \frac{K_1}{2} 2^{2q} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{\bar{n}} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \right) \right. \\
& \left. + \frac{K_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi \right\}^{1/2}
\end{aligned}$$

after eliminating the quadratic terms, then multiplying the factor $2^{q(\sigma-1)}$ on the both sides of inequality, we arrive at (4.29) immediately with the help of Lemma 4.6 and the smallness of ϵ ($0 < \epsilon \leq 1$). \square

Similarly, we also have the *a priori* estimate for the case of low frequency.

Lemma 4.8. ($q = -1$) *There exist some positive constants $\bar{K}_1, \bar{K}_2, \bar{K}_3$ and μ_3 independent of ϵ , such that the following estimate holds:*

$$\begin{aligned}
& 2^{-(\sigma-1)} \frac{d}{dt} \left\{ \frac{\bar{K}_1}{2} 2^{-2} \left(\|\Delta_{-1} m^\epsilon\|_{L^2}^2 + \|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{n} \|\Delta_{-1} \nabla \phi^\epsilon\|_{L^2}^2 \right) \right. \\
& + \frac{\bar{K}_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_{-1} W^\epsilon})^* K(\xi) \widehat{\Delta_{-1} W^\epsilon} \right) d\xi - \bar{K}_3 \epsilon \int \Delta_{-1} \nabla \phi^\epsilon \cdot \overline{\Delta_{-1} \mathbf{v}^\epsilon} dx \left. \right\}^{1/2} \\
& + \mu_3 2^{-\sigma} \left(\|\Delta_{-1} m^\epsilon\|_{L^2} + \|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2} + \|\Delta_{-1} \nabla \phi^\epsilon\|_{L^2} \right) \\
& \leq C 2^{-\sigma} \|(W^\epsilon, \nabla W^\epsilon)\|_{L^\infty} \|\Delta_{-1} W^\epsilon\|_{L^2} + C c_{-1} \|W^\epsilon\|_{B_{2,1}^\sigma}^2 \\
& + \frac{C}{\epsilon} 2^{-\sigma} \|\Delta_{-1} (h(\epsilon m^\epsilon) \mathbf{v}^\epsilon)\|_{L^2} + C 2^{-(\sigma-1)} \|\Delta_{-1} \mathcal{G}\|_{L^2} \\
& + C 2^{-\sigma} \|\Delta_{-1} (H(\epsilon m^\epsilon) m^\epsilon)\|_{L^2}, \tag{4.34}
\end{aligned}$$

where C is a uniform positive constant independent of ϵ .

Remark 4.1. Similar to the proof of Lemma 4.7, the constants $\bar{K}_1, \bar{K}_2, \bar{K}_3$ are introduced to ensure that

$$\begin{aligned}
& \frac{\bar{K}_1}{2} 2^{-2} \left(\|\Delta_{-1} m^\epsilon\|_{L^2}^2 + \|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{n} \|\Delta_{-1} \nabla \phi^\epsilon\|_{L^2}^2 \right) \\
& + \frac{\bar{K}_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_{-1} W^\epsilon})^* K(\xi) \widehat{\Delta_{-1} W^\epsilon} \right) d\xi - \bar{K}_3 \epsilon \int \Delta_{-1} \nabla \phi^\epsilon \cdot \overline{\Delta_{-1} \mathbf{v}^\epsilon} dx \\
& \approx 2^{-2} \left(\|\Delta_{-1} m^\epsilon\|_{L^2}^2 + \|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2}^2 + \|\Delta_{-1} \nabla \phi^\epsilon\|_{L^2}^2 \right). \tag{4.35}
\end{aligned}$$

and eliminate some quadratic terms appearing in the right-hand side of inequality (4.34).

Summing (4.29) on $q \in \mathbb{N} \cup \{0\}$ and adding (4.34) together, according to the *a priori* assumption (4.1), (4.28) and Moser's estimates (Proposition 2.4), we obtain the following differential inequality:

$$\frac{d}{dt} Q(t) + \mu_4 \|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{B_{2,1}^\sigma} \leq C \delta_1 \|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{B_{2,1}^\sigma}, \tag{4.36}$$

where

$$\begin{aligned}
Q(t) &= \sum_{q \geq 0} 2^{q(\sigma-1)} \left\{ \frac{K_1}{2} 2^{2q} \left(\|\Delta_q m^\epsilon\|_{L^2}^2 + \|\Delta_q \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{n} \|\Delta_q \nabla \phi^\epsilon\|_{L^2}^2 \right) \right. \\
& + \frac{K_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_q W^\epsilon})^* K(\xi) \widehat{\Delta_q W^\epsilon} \right) d\xi \left. \right\}^{1/2} \\
& + \left\{ \frac{\bar{K}_1}{2} 2^{-2} \left(\|\Delta_{-1} m^\epsilon\|_{L^2}^2 + \|\Delta_{-1} \mathbf{v}^\epsilon\|_{L^2}^2 + \frac{1}{n} \|\Delta_{-1} \nabla \phi^\epsilon\|_{L^2}^2 \right) \right. \\
& + \frac{\bar{K}_2 \epsilon}{2} \operatorname{Im} \int |\xi| \left((\widehat{\Delta_{-1} W^\epsilon})^* K(\xi) \widehat{\Delta_{-1} W^\epsilon} \right) d\xi \\
& \left. - \bar{K}_3 \int \Delta_{-1} \nabla \phi^\epsilon \cdot \overline{\Delta_{-1} \mathbf{v}^\epsilon} dx \right\}^{1/2} \tag{4.37}
\end{aligned}$$

and μ_4 is some positive constant. Note that

$$Q(t) \approx \|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)(\cdot, t)\|_{B_{2,1}^\sigma}, \quad t \geq 0, \tag{4.38}$$

and by choosing $\delta_1 = \min\{\frac{\mu_4}{2C}, \frac{\bar{\psi}}{(\gamma-1)C}\}$, we get

$$\|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)(\cdot, t)\|_{B_{2,1}^\sigma} \leq C \left\| \left(\frac{m_0}{\epsilon}, \mathbf{v}_0, \frac{\mathbf{e}_0}{\epsilon} \right) \right\|_{B_{2,1}^\sigma} \exp(-\mu_1 t), \tag{4.39}$$

where we have used the Gronwall's inequality, $\mu_1 := \frac{\mu_4}{2}$. This is just the inequality (4.2).

Hence, the proof of Proposition 4.1 is complete. \square

5 Zero-electron-mass limit

In this section, our first aim at deducing a frequency-localization Strichartz-type estimate which is the crucial ingredient of the zero-electron-mass limit. For this end, we need to give a detailed analysis of the equation of acoustics (5.3) with the aid of the semigroup formulation, and obtain some dispersive estimates. Then, by using the classical TT^* argument, we obtain the desired Strichartz-type estimate.

5.1 The linearized system

From (1.9), we have

$$\begin{cases} m_t^\epsilon + \bar{\psi} \frac{\operatorname{div} \mathbf{v}^\epsilon}{\epsilon} = F, \\ \mathbf{v}_t^\epsilon + \mathbf{v}^\epsilon + \bar{\psi} \frac{\nabla m^\epsilon}{\epsilon} - \frac{h'(0) \nabla \Delta^{-1} m^\epsilon}{\epsilon} = G, \end{cases} \quad (5.1)$$

where $F = -\mathbf{v}^\epsilon \cdot \nabla m^\epsilon - \frac{\gamma-1}{2} m^\epsilon \operatorname{div} \mathbf{v}^\epsilon$, $G = -\mathbf{v}^\epsilon \cdot \nabla \mathbf{v}^\epsilon - \frac{\gamma-1}{2} m^\epsilon \nabla m^\epsilon + \frac{\nabla \Delta^{-1} (h(\epsilon m^\epsilon) - h'(0) \epsilon m^\epsilon)}{\epsilon^2}$. Following from the idea in [4], we split the velocity into a divergence-free part $\mathcal{P} \mathbf{v}^\epsilon$ and a gradient part $\mathcal{Q} \mathbf{v}^\epsilon$, where $\mathcal{P} = I - \nabla \Delta^{-1} \operatorname{div}$ and $\mathcal{Q} = I - \mathcal{P} = \nabla \Delta^{-1} \operatorname{div}$. Then, the system (5.1) reads

$$\begin{cases} m_t^\epsilon + \bar{\psi} \frac{\operatorname{div} \mathcal{Q} \mathbf{v}^\epsilon}{\epsilon} = F, \\ \mathcal{Q} \mathbf{v}_t^\epsilon + \mathcal{Q} \mathbf{v}^\epsilon + \bar{\psi} \frac{\nabla m^\epsilon}{\epsilon} - \frac{h'(0) \nabla \Delta^{-1} m^\epsilon}{\epsilon} = \mathcal{Q} G, \\ \mathcal{P} \mathbf{v}_t^\epsilon + \mathcal{P} \mathbf{v}^\epsilon = \mathcal{P} G, \\ \mathbf{v}^\epsilon = \mathcal{P} \mathbf{v}^\epsilon + \mathcal{Q} \mathbf{v}^\epsilon. \end{cases} \quad (5.2)$$

Because of $\|\mathcal{Q} \mathbf{v}^\epsilon\|_{B_{2,1}^\sigma} \approx \|d^\epsilon\|_{B_{2,1}^\sigma}$, where $d^\epsilon = \Lambda^{-1} \operatorname{div} \mathcal{Q} \mathbf{v}^\epsilon$ with $\mathcal{F}(\Lambda^{-1} f) = |\xi|^{-1} \mathcal{F} f$, we shall investigate carefully the following mixed linear equation of acoustics:

$$\begin{cases} m_t^\epsilon + \bar{\psi} \frac{\Lambda d^\epsilon}{\epsilon} = F, \\ d_t^\epsilon + d^\epsilon - \bar{\psi} \frac{\Lambda m^\epsilon}{\epsilon} - \frac{h'(0) \Lambda^{-1} m^\epsilon}{\epsilon} = \Lambda^{-1} \operatorname{div} \mathcal{Q} G, \\ (m^\epsilon, d^\epsilon)|_{t=0} = (m_0^\epsilon, d_0^\epsilon), \end{cases} \quad (5.3)$$

where $d_0^\epsilon := \Lambda^{-1} \operatorname{div} \mathcal{Q} \mathbf{v}_0^\epsilon$.

According to the semigroup theory for evolutionary equation, the solutions (m^ϵ, d^ϵ) to the linear initial value problem (5.3) can be expressed for $U^\epsilon = (m^\epsilon, d^\epsilon)^\top$ as

$$U_t^\epsilon = B U^\epsilon + (F, \Lambda^{-1} \operatorname{div} \mathcal{Q} G)^\top, \quad U^\epsilon(0) = U_0^\epsilon = (m_0^\epsilon, d_0^\epsilon)^\top, \quad t \geq 0, \quad (5.4)$$

which gives rise to

$$U^\epsilon(t) = S(t) U_0^\epsilon + \int_0^t S(t-\tau) (F, \Lambda^{-1} \operatorname{div} \mathcal{Q} G)^\top d\tau, \quad t \geq 0, \quad (5.5)$$

where $S(t) U_0^\epsilon := e^{tB} U_0^\epsilon$. Then, we analyze the differential operator B by means of its Fourier expression $A(\xi)$ and show the long time properties of the semigroup $S(t)$. Applying the Fourier transform to the system (5.4) with $F = 0$ and $G = 0$, we get

$$\partial_t \widehat{U}^\epsilon = A(\xi) \widehat{U}^\epsilon, \quad \widehat{U}^\epsilon(0) = \widehat{U}_0^\epsilon, \quad (5.6)$$

where $\widehat{U}^\epsilon(t) = \widehat{U}^\epsilon(\xi, t) = \mathcal{F} U^\epsilon(\xi, t)$, $\xi = (\xi_1, \dots, \xi_N)^\top$ and $A(\xi)$ is defined as

$$A(\xi) = \begin{pmatrix} 0 & -\frac{\bar{\psi} |\xi|}{\epsilon} \\ \frac{\bar{\psi} |\xi|}{\epsilon} + \frac{h'(0)}{|\xi| \epsilon} & -1 \end{pmatrix}. \quad (5.7)$$

The eigenvalues of the matrix $A(\xi)$ are computed from the determinant

$$\det(A(\xi) - \lambda I) = \begin{vmatrix} -\lambda & -\frac{\bar{\psi} |\xi|}{\epsilon} \\ \frac{\bar{\psi} |\xi|}{\epsilon} + \frac{h'(0)}{|\xi| \epsilon} & -1 - \lambda \end{vmatrix} = 0,$$

which implies

$$\lambda_{\pm} = -\frac{1}{2} \pm \frac{i}{\epsilon} \sqrt{\bar{\psi}^2 |\xi|^2 + \bar{\psi} h'(0) - \frac{1}{4} \epsilon^2} := -\frac{1}{2} \pm \frac{i}{\epsilon} \lambda.$$

Hence, the semigroup e^{tA} exhibits the expression

$$e^{tA} = e^{\lambda_+ t} \frac{A - \lambda_- I}{\lambda_+ - \lambda_-} + e^{\lambda_- t} \frac{A - \lambda_+ I}{\lambda_- - \lambda_+} := e^{\lambda_+ t} P_+ + e^{\lambda_- t} P_-.$$

After a direct computation, we can verify the exact expression about the Fourier transform $\widehat{e^{tB}}$ of the Green's function e^{tB} as

$$\begin{aligned} \widehat{e^{tB}} &:= e^{tA} = e^{\lambda_+ t} P_+ + e^{\lambda_- t} P_- \\ &= \begin{pmatrix} \frac{-\lambda_- e^{\lambda_+ t} + \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} & -\frac{\bar{\psi} |\xi|}{\epsilon} \cdot \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \\ \left(\frac{\bar{\psi} |\xi|}{\epsilon} + \frac{h'(0)}{|\xi| \epsilon} \right) \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} & -\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} + \frac{-\lambda_- e^{\lambda_+ t} + \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{pmatrix}. \end{aligned} \quad (5.8)$$

Lemma 5.1 (Dispersive estimate). *With the above notations, when ϵ is small enough, we get the following estimate:*

$$|I_{1,k}(t, \tau, z)| \leq \epsilon C 2^{(N-1)k} \max\{1, 2^{-k}\} e^{-\frac{1}{2}t} \min\left\{\frac{2^k}{2^k + 1}, \tau^{-\frac{1}{2}}\right\}, \quad (5.9)$$

where

$$I_{1,k}(t, \tau, z) = \int_{\mathbb{R}^N} e^{i\xi \cdot z} \psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t} e^{i\tau\lambda}}{\lambda_+ - \lambda_-} d\xi,$$

and smooth function ψ satisfies $\text{supp} \psi(x) \in \{x \in \mathbb{R} | \frac{1}{6} \leq x \leq 3\}$ and $\psi(x)|_{\frac{5}{8} \leq |x| \leq \frac{12}{5}} = 1$, $C > 0$ denotes a uniform constant independent of k and ϵ .

Proof. Using the rotation invariant in ξ , we restrict ourselves to the case when $z_2 = \dots = z_N = 0$. The estimate will follow from the stationary phase theorem. Denoting $\alpha(\xi) := -\partial_{\xi_2}(\lambda) = -\frac{\bar{\psi}^2 \xi_2}{\lambda}$, we introduce the following differential operator

$$\mathcal{L} := \frac{1 + i\alpha(\xi) \partial_{\xi_2}}{1 + \tau \alpha^2(\xi)},$$

which acts on the ξ_2 variable, and satisfies $\mathcal{L}(e^{i\tau\lambda}) = e^{i\tau\lambda}$. Easy computation yields

$${}^{\top} \mathcal{L} = \frac{1}{1 + \tau \alpha^2(\xi)} - i(\partial_{\xi_2} \alpha) \frac{1 - \tau \alpha^2}{(1 + \tau \alpha^2)^2} - \frac{i\alpha}{1 + \tau \alpha^2} \partial_{\xi_2}.$$

Using the integration by parts, we obtain

$$I_{1,k}(t, \tau, z) = \int_{\mathbb{R}^N} {}^{\top} \mathcal{L} \left[\psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t}}{\lambda_+ - \lambda_-} \right] e^{i\tau\lambda + iz_1 \xi_1} d\xi,$$

where

$$\begin{aligned} &{}^{\top} \mathcal{L} \left[\psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t}}{\lambda_+ - \lambda_-} \right] \\ &= \left(\frac{1}{1 + \tau \alpha^2(\xi)} - i(\partial_{\xi_2} \alpha) \frac{1 - \tau \alpha^2}{(1 + \tau \alpha^2)^2} \right) \psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t}}{\lambda_+ - \lambda_-} \\ &\quad - \frac{i\alpha}{1 + \tau \alpha^2} \partial_{\xi_2} \left(\psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t}}{\lambda_+ - \lambda_-} \right). \end{aligned}$$

Because of $\frac{1}{6} 2^k \leq |\xi| \leq 3 \cdot 2^k$, when ϵ is small enough satisfying

$$\bar{\psi}^2 |\xi|^2 - \frac{1}{8} \epsilon^2 > \frac{1}{2} \bar{\psi}^2 |\xi|^2, \quad \bar{\psi} h'(0) - \frac{1}{8} \epsilon^2 > \frac{1}{2} \bar{\psi} h'(0) \text{ and } 2^k \epsilon \leq 1,$$

we have

$$\begin{aligned}
|\alpha| &\leq C, \\
\frac{1}{1+\tau\alpha^2} + \frac{|\alpha|}{1+\tau\alpha^2} &\leq \frac{C}{1+(2^k+1)^{-2}\tau\xi_2^2}, \\
\frac{|\partial_{\xi_2}\alpha||1-\tau\alpha^2|}{(1+\tau\alpha^2)^2} &\leq \frac{C(2^k+1)^{-1}}{1+(2^k+1)^{-2}\tau\xi_2^2}, \\
\left| \psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t}}{\lambda_+ - \lambda_-} \right| &\leq C\epsilon(2^k+1)^{-1}e^{-\frac{1}{2}t}, \\
|\partial_{\xi_2}(\psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t}}{\lambda_+ - \lambda_-})| &\leq C\epsilon 2^{-k}(2^k+1)^{-1}e^{-\frac{1}{2}t}.
\end{aligned}$$

An easy computation shows that

$$\begin{aligned}
&|I_{1,k}(t, \tau, z)| \\
&\leq \epsilon C(2^k+1)^{-1} \max\{1, 2^{-k}\} e^{-\frac{1}{2}t} \int_{\frac{1}{8}2^k \leq |\xi| \leq 3 \cdot 2^k} \frac{d\xi}{1+(2^k+1)^{-2}\tau\xi_2^2} \\
&\leq \epsilon C 2^{(N-1)k} \max\{1, 2^{-k}\} e^{-\frac{1}{2}t} \min\left\{\frac{2^k}{2^k+1}, \tau^{-\frac{1}{2}}\right\}.
\end{aligned}$$

Hence the estimate (5.9) holds. \square

Similarly, we can obtain the following lemma and omit the proofs for brevity.

Lemma 5.2 (Dispersive estimates). *With the above notations, when ϵ is small enough, we get the following estimates:*

$$|I_{2,k}(t, \tau, z)| \leq \epsilon C 2^{(N-1)k} \max\{1, 2^{-k}\} e^{-\frac{1}{2}t} \min\left\{\frac{2^k}{2^k+1}, \tau^{-\frac{1}{2}}\right\}, \quad (5.10)$$

$$|I_{3,k}(t, \tau, z)| \leq C 2^{Nk} \max\{1, 2^{-2k}\} e^{-\frac{1}{2}t} \min\left\{\frac{2^k}{2^k+1}, \tau^{-\frac{1}{2}}\right\}, \quad (5.11)$$

$$|I_{4,k}(t, \tau, z)| \leq C 2^{Nk} \max\{1, 2^{-k}\} e^{-\frac{1}{2}t} \min\left\{\frac{2^k}{2^k+1}, \tau^{-\frac{1}{2}}\right\}, \quad (5.12)$$

$$|I_{5,k}(t, \tau, z)| \leq C 2^{(N-2)k} \max\{1, 2^{-k}\} e^{-\frac{1}{2}t} \min\left\{\frac{2^k}{2^k+1}, \tau^{-\frac{1}{2}}\right\}, \quad (5.13)$$

where

$$\begin{aligned}
I_{2,k}(t, \tau, z) &= \int_{\mathbb{R}^N} e^{i\xi \cdot z} \psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t} e^{-i\tau\lambda}}{\lambda_+ - \lambda_-} d\xi, \\
I_{3,k}(t, \tau, z) &= \int_{\mathbb{R}^N} e^{i\xi \cdot z} \psi(2^{-k}|\xi|) \frac{\lambda_{\mp} e^{-\frac{1}{2}t} e^{\pm i\tau\lambda}}{\lambda_+ - \lambda_-} d\xi, \\
I_{4,k}(t, \tau, z) &= \int_{\mathbb{R}^N} e^{i\xi \cdot z} \psi(2^{-k}|\xi|) \frac{\bar{\psi}|\xi|}{\epsilon} \frac{e^{-\frac{1}{2}t} e^{\pm i\tau\lambda}}{\lambda_+ - \lambda_-} d\xi, \\
I_{5,k}(t, \tau, z) &= \int_{\mathbb{R}^N} e^{i\xi \cdot z} \psi(2^{-k}|\xi|) \frac{h'(0)}{|\xi|\epsilon} \frac{e^{-\frac{1}{2}t} e^{\pm i\tau\lambda}}{\lambda_+ - \lambda_-} d\xi,
\end{aligned}$$

and $C > 0$ denotes a uniform constant independent of k and ϵ .

Proposition 5.3 (Strichartz-type estimate). *Suppose U^ϵ is the solution of the system (5.4) with*

$$\text{supp } \hat{U}_0 \cup \left\{ \bigcup_{t \geq 0} \text{supp}(\hat{F}(t), \hat{G}(t))^\top \right\} \in \mathcal{C}_k = \left\{ \xi \in \mathbb{R}^N \mid \frac{5}{6} 2^k \leq |\xi| \leq \frac{12}{5} \cdot 2^k \right\}.$$

When ϵ is small enough, we get the following estimate

$$\|U^\epsilon\|_{L^1(\mathbb{R}^+; L_x^p)} \leq \begin{cases} C 2^{\frac{Nk(p-2)}{2p}} \epsilon^{\frac{p-2}{4p}} (\|U_0^\epsilon\|_{L^2} + \|(F, G)\|_{L^1(\mathbb{R}^+; L^2)}) & k \geq 0, \\ C 2^{\frac{(N-4)k(p-2)}{2p}} \epsilon^{\frac{p-2}{4p}} (\|U_0^\epsilon\|_{L^2} + \|(F, G)\|_{L^1(\mathbb{R}^+; L^2)}) & k < 0, \end{cases} \quad (5.14)$$

where $p \in [2, +\infty]$.

Proof. Duhamel's formula enable us to restrict our attentions to the case $(F, G) = 0$. if we can obtain

$$\|S(t)U_0^\epsilon\|_{L^1(\mathbb{R}^+; L_x^p)} \leq \begin{cases} C 2^{\frac{Nk(p-2)}{2p}} \epsilon^{\frac{p-2}{4p}} \|U_0^\epsilon\|_{L^2} & k \geq 0, \\ C 2^{\frac{(N-4)k(p-2)}{2p}} \epsilon^{\frac{p-2}{4p}} \|U_0^\epsilon\|_{L^2} & k < 0, \end{cases} \quad (5.15)$$

by the same computations as that in [2], we can get the estimate on

$$\int_0^t S(t-\tau)(F, \Lambda^{-1} \text{div } \mathcal{Q}G)^\top(\tau) d\tau$$

by Fubini theorem and interpolation. Finally, we can get (5.14).

In the following, we mainly prove the estimate (5.15). For simplicity, we only estimate the term

$$A_{1,k} d_0^\epsilon = \int_{\mathbb{R}^N} I_{1,k}(t, \tau, x-y) d_0^\epsilon(y) dy,$$

where

$$I_{1,k}(t, \tau, z) = \int_{\mathbb{R}^N} e^{i\xi \cdot z} \psi(2^{-k}|\xi|) \frac{e^{-\frac{1}{2}t} e^{i\tau\lambda}}{\lambda_+ - \lambda_-} d\xi,$$

$\tau = \frac{t}{\epsilon}$, and ψ satisfies $\text{supp } \psi(x) \in \{x \in \mathbb{R} \mid \frac{1}{6} \leq x \leq 3\}$ and $\psi(x)|_{\frac{5}{6} \leq |x| \leq \frac{12}{5}} = 1$.

Now we shall use the TT^* argument. Define

$$B := \{a \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^N), \|a\|_{L^\infty(\mathbb{R}^+; L^1)} \leq 1\},$$

we have

$$\begin{aligned} \|A_{1,k} d_0^\epsilon\|_{L^1(\mathbb{R}^+; L_x^\infty)} &= \sup_{a \in B} \int_{\mathbb{R}^+ \times \mathbb{R}^{2N}} I_{1,k}(t, \tau, x-y) d_0^\epsilon(y) a(t, x) dt dx dy \\ &\leq \|d_0^\epsilon\|_{L^2} \sup_{a \in B} \left\| \int_{\mathbb{R}^+} \check{I}_{1,k}(t, \tau, \cdot) * a(t, \cdot) dt \right\|_{L^2}, \end{aligned}$$

where $\check{b}(x) = b(-x)$. Let

$$\Phi := \left\| \int_{\mathbb{R}^+} \check{I}_{1,k}(t, \tau, \cdot) * a(t, \cdot) dt \right\|_{L^2}.$$

By the Fourier-Planchered theorem, we have

$$\Phi^2 = C \int_{(\mathbb{R}^+)^2 \times \mathbb{R}^N} \hat{I}_{1,k}(t, \tau, -\xi) \hat{a}(t, \xi) \bar{\hat{I}}_{1,k}(s, \frac{s}{\epsilon}, -\xi) \bar{\hat{a}}(s, \xi) dt ds d\xi.$$

Note that the following identity holds

$$\hat{I}_{1,k}(t, \tau, -\xi) \bar{\hat{I}}_{1,k}(s, \frac{s}{\epsilon}, -\xi) = \hat{I}_{1,k}(t+s, \frac{t-s}{\epsilon}, -\xi) \frac{\psi(2^{-k}|\xi|)}{\lambda_+ - \lambda_-},$$

from Lemma 5.1, it follows that

$$\begin{aligned}
& \Phi^2 \\
&= C \int_{(\mathbb{R}^+)^2 \times \mathbb{R}^N} \frac{\psi(2^{-k}|\xi|)}{\lambda_+ - \lambda_-} \mathcal{F}(\check{I}_{1,k}(t+s, \frac{t-s}{\epsilon}, \cdot) * a(t, \cdot)) \bar{\hat{a}}(s, \xi) dt ds d\xi \\
&= C \int_{(\mathbb{R}^+)^2 \times \mathbb{R}^N} (\check{I}_{1,k}(t+s, \frac{t-s}{\epsilon}, \cdot) * a(t, \cdot))(x) \mathcal{F}^{-1} \left\{ \frac{\psi(2^{-k}|\xi|)}{\lambda_+ - \lambda_-} \hat{a} \right\}(s, x) dt ds dx \\
&\leq C \epsilon (2^k + 1)^{-1} \int_{(\mathbb{R}^+)^2} \|\check{I}_{1,k}(t+s, \frac{t-s}{\epsilon}, \cdot) * a(t, \cdot)\|_{L_x^\infty} \|a(s, \cdot)\|_{L_x^1} dt ds \\
&\leq C \epsilon (2^k + 1)^{-1} \int_{(\mathbb{R}^+)^2} \|\check{I}_{1,k}(t+s, \frac{t-s}{\epsilon}, \cdot)\|_{L_x^\infty} \|a(t, \cdot)\|_{L_x^1} \|a(s, \cdot)\|_{L_x^1} dt ds \\
&\leq \epsilon^2 C 2^{(N-2)k} \int_{(\mathbb{R}^+)^2} \min \left\{ \frac{2^k}{2^k + 1}, \frac{\epsilon^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}} \right\} e^{-\frac{1}{2}(t+s)} dt ds \\
&\leq C 2^{(N-2)k} \epsilon^{\frac{5}{2}}.
\end{aligned}$$

Then we have

$$\|A_{1,k} d_0^\epsilon\|_{L^1(\mathbb{R}^+; L^\infty)} \leq C \epsilon^{\frac{5}{4}} 2^{\frac{(N-2)k}{2}} \|d_0^\epsilon\|_{L^2}.$$

Similarly, from Lemma 5.2, we can estimate the other terms in $S(t)U_0^\epsilon$ and obtain

$$\|U^\epsilon\|_{L^1(\mathbb{R}^+; L^\infty)} \leq \begin{cases} C 2^{\frac{Nk}{2}} \epsilon^{\frac{1}{4}} \|U_0^\epsilon\|_{L^2} & k \geq 0, \\ C 2^{\frac{(N-4)k}{2}} \epsilon^{\frac{1}{4}} \|U_0^\epsilon\|_{L^2} & k < 0. \end{cases}$$

Similarly, we can obtain that the $L^1(\mathbb{R}^+; L^2(\mathbb{R}^N))$ norm of U^ϵ is bounded, uniformly in ϵ , by interpolation, we get (5.15). \square

5.2 Global convergence

Proposition 5.4. *Suppose $(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)$ is the solution of the system (1.9)-(1.10), then we have*

$$\|(m^\epsilon, \mathcal{Q}\mathbf{v}^\epsilon)\|_{L^1(\mathbb{R}^+; B_{p,1}^{\frac{N}{p}})} \leq \begin{cases} C \epsilon^{\frac{(p-2)^2 N}{8p(3p-4)}}, & 1 - \frac{(p-2)(N-4)}{2p} > 0, \\ C_\beta \epsilon^{\frac{p-2-\beta}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} = 0, \\ C \epsilon^{\frac{p-2}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} < 0, \end{cases} \quad (5.16)$$

$$\|\nabla \phi^\epsilon\|_{L^1(\mathbb{R}^+; B_{p,1}^{\frac{N}{p}})} \leq \begin{cases} C \epsilon^{\frac{(p-2)^2 N}{32p(p-1)}}, & 2 - \frac{(p-2)(N-4)}{2p} > 0, \\ C_\beta \epsilon^{\frac{p-2-\beta}{4p}}, & 2 - \frac{(p-2)(N-4)}{2p} = 0, \\ C \epsilon^{\frac{p-2}{4p}}, & 2 - \frac{(p-2)(N-4)}{2p} < 0, \end{cases} \quad (5.17)$$

where $p \in [2, \infty]$ and $\beta \in (0, p-2)$.

Proof. From Proposition 4.1, we get $m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon$ uniformly bounded in $L_t^\infty B_{2,1}^\sigma \cap L_t^1 B_{2,1}^\sigma$. Let $P_{\geq M} f = \mathcal{F}^{-1}(\chi_{|\xi| \geq M} \hat{f})$, then we have

$$\begin{aligned}
\|P_{\geq M}(m^\epsilon, \mathcal{Q}\mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{L_t^1 B_{p,1}^{\frac{N}{p}}} &\leq C \|P_{\geq M}(m^\epsilon, \mathcal{Q}\mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{L_t^1 B_{2,1}^{\frac{N}{2}}} \\
&\leq C M^{-1} \|P_{\geq M}(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{L_t^1 B_{2,1}^\sigma},
\end{aligned} \quad (5.18)$$

where $p \in [2, \infty]$. Let $P_{\leq \frac{1}{M}} f = \mathcal{F}^{-1}(\chi_{|\xi| \leq \frac{1}{M}} \hat{f})$, then we obtain

$$\begin{aligned}
\|P_{\leq \frac{1}{M}}(m^\epsilon, \mathcal{Q}\mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{L_t^1 L^p} &\leq C J^{-\frac{(p-2)N}{2p}} \|P_{\leq \frac{1}{M}}(m^\epsilon, \mathcal{Q}\mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{L_t^1 L^2} \\
&\leq C J^{-\frac{(p-2)N}{2p}} \|(m^\epsilon, \mathbf{v}^\epsilon, \nabla \phi^\epsilon)\|_{L_t^1 L^2}.
\end{aligned} \quad (5.19)$$

where $p \in [2, \infty]$.

From (5.14), we have

$$\begin{aligned} & \|\dot{\Delta}_k(m^\epsilon, d^\epsilon)\|_{L^1(\mathbb{R}^+; L^p)} \\ & \leq \begin{cases} C 2^{\frac{Nk(p-2)}{2p}} \epsilon^{\frac{p-2}{4p}} \|\dot{\Delta}_k(m_0, d_0)\|_{L^2} + \|\dot{\Delta}_k(F, G)\|_{L^1(\mathbb{R}^+; L^2)} & k \geq 0, \\ C 2^{\frac{(N-4)k(p-2)}{2p}} \epsilon^{\frac{p-2}{4p}} \|\dot{\Delta}_k(m_0, d_0)\|_{L^2} + \|\dot{\Delta}_k(F, G)\|_{L^1(\mathbb{R}^+; L^2)} & k < 0, \end{cases} \end{aligned} \quad (5.20)$$

where $p \in [2, +\infty]$. From Theorem 1.1 and the classical estimate in Besov space for the product of two functions $B_{2,1}^{\frac{N}{2}} \times B_{2,1}^{\frac{N}{2}} \hookrightarrow B_{2,1}^{\frac{N}{2}}$, we have

$$\|\dot{\Delta}_k(F, G)\|_{L^1(\mathbb{R}^+; L^2)} \leq C c_k 2^{-\frac{Nk}{2}}, \quad k \geq 0,$$

and

$$\|\dot{\Delta}_k(F, G)\|_{L^1(\mathbb{R}^+; L^2)} \leq C 2^{-k}, \quad k < 0,$$

where C is independent of ϵ and $\sum c_k \leq 1$. Let $P_{A \leq \cdot \leq B} f = \mathcal{F}^{-1}(\chi_{A \leq |\xi| \leq B} \hat{f})$. Then, we have

$$\|P_{\frac{1}{J} \leq \cdot \leq 1}(m^\epsilon, d^\epsilon)\|_{L^1(\mathbb{R}^+; L^p)} \leq \begin{cases} C J^{1 - \frac{(p-2)(N-4)}{2p}} \epsilon^{\frac{p-2}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} > 0, \\ C \ln J \epsilon^{\frac{p-2}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} = 0, \\ C \epsilon^{\frac{p-2}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} < 0, \end{cases} \quad (5.21)$$

and

$$\|P_{1 \leq \cdot \leq M}(m^\epsilon, Q\mathbf{v}^\epsilon)\|_{L^1(\mathbb{R}^+; B_{p,1}^{\frac{N}{p}})} \leq C \epsilon^{\frac{p-2}{4p}}, \quad (5.22)$$

where $p \in [2, \infty]$. From (5.18)–(5.19) and (5.21)–(5.22), we have

$$\|(m^\epsilon, Q\mathbf{v}^\epsilon)\|_{L^1(\mathbb{R}^+; B_{p,1}^{\frac{N}{p}})} \leq \begin{cases} C \epsilon^{\frac{(p-2)^2 N}{8p(3p-4)}}, & 1 - \frac{(p-2)(N-4)}{2p} > 0, \\ C_\beta \epsilon^{\frac{p-2-\beta}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} = 0, \\ C \epsilon^{\frac{p-2}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} < 0, \end{cases} \quad (5.23)$$

where $p \in [2, \infty]$, $\beta \in (0, p-2)$, $J = \epsilon^{-\frac{p-2}{4(3p-4)}}$ and $M = \epsilon^{-\frac{(p-2)^2 N}{8p(3p-4)}}$ when $1 - \frac{(p-2)(N-4)}{2p} > 0$, $J = \epsilon^{-\frac{p-2-\beta}{2N(p-2)}}$ and $M = \epsilon^{-\frac{p-2-\beta}{4p}}$ when $1 - \frac{(p-2)(N-4)}{2p} = 0$, $J = \epsilon^{-\frac{1}{2N}}$ and $M = \epsilon^{-\frac{p-2}{4p}}$ when $1 - \frac{(p-2)(N-4)}{2p} < 0$.

Similarly, we have

$$\|P_{\frac{1}{J} \leq \cdot \leq 1} \nabla \phi^\epsilon\|_{L^1(\mathbb{R}^+; L^p)} \leq \begin{cases} C J^{2 - \frac{(p-2)(N-4)}{2p}} \epsilon^{\frac{p-2}{4p}}, & 2 - \frac{(p-2)(N-4)}{2p} > 0, \\ C \ln J \epsilon^{\frac{p-2}{4p}}, & 2 - \frac{(p-2)(N-4)}{2p} = 0, \\ C \epsilon^{\frac{p-2}{4p}}, & 2 - \frac{(p-2)(N-4)}{2p} < 0, \end{cases} \quad (5.24)$$

$$\|P_{1 \leq \cdot \leq M} \nabla \phi^\epsilon\|_{L^1(\mathbb{R}^+; B_{p,1}^{\frac{N}{p}})} \leq C \epsilon^{\frac{p-2}{4p}}, \quad (5.25)$$

and

$$\|\nabla \phi^\epsilon\|_{L^1(\mathbb{R}^+; B_{p,1}^{\frac{N}{p}})} \leq \begin{cases} C \epsilon^{\frac{(p-2)^2 N}{32p(p-1)}}, & 2 - \frac{(p-2)(N-4)}{2p} > 0, \\ C_\beta \epsilon^{\frac{p-2-\beta}{4p}}, & 2 - \frac{(p-2)(N-4)}{2p} = 0, \\ C \epsilon^{\frac{p-2}{4p}}, & 2 - \frac{(p-2)(N-4)}{2p} < 0, \end{cases} \quad (5.26)$$

where $p \in [2, \infty]$ and $\beta \in (0, p-2)$. \square

Now, we consider the global well-posedness of incompressible Euler equations (1.4). Following from the standard frequency-localization method, we can easily obtain the following result.

Theorem 5.5. *There exists $\delta > 0$, such that if $\|\mathcal{P}\mathbf{v}_0\|_{B_{2,1}^\sigma} \leq \delta$, then there exists a unique solution \mathbf{u} to the incompressible Euler equations (1.4) satisfying*

$$\|\mathbf{u}\|_{L^\infty(\mathbb{R}^+; B_{2,1}^\sigma) \cap L^1(\mathbb{R}^+; B_{2,1}^\sigma)} \leq C\|\mathcal{P}\mathbf{v}_0\|_{B_{2,1}^\sigma}. \quad (5.27)$$

Proposition 5.6. *Suppose $(m^\epsilon, \mathbf{v}^\epsilon, \nabla\phi^\epsilon)$ is the solution of the system (1.9)-(1.10), then*

$$\begin{aligned} & \|\mathcal{P}\mathbf{v}^\epsilon - \mathbf{u}\|_{L^\infty([0,T]; B_{p,1}^{\frac{N}{p}}) \cap L^1([0,T]; B_{p,1}^{\frac{N}{p}})} \\ & \leq \begin{cases} C\epsilon^{\frac{(p-2)^2 N}{4p(3p-4)(N+2)}}, & 1 - \frac{(p-2)(N-4)}{2p} > 0 \text{ and } 2 \leq p \leq N, \\ C\epsilon^{\frac{(p-2)^2 N^2}{4p^2(3p-4)(N+2)}}, & 1 - \frac{(p-2)(N-4)}{2p} > 0 \text{ and } N < p \leq \infty, \\ C\beta\epsilon^{\frac{p-2-\beta}{2p(N+2)}}, & 1 - \frac{(p-2)(N-4)}{2p} = 0, \beta \in (0, p-2), \\ C\epsilon^{\frac{p-2}{2p(N+2)}}, & 1 - \frac{(p-2)(N-4)}{2p} < 0 \text{ and } 2 \leq p \leq N, \\ C\epsilon^{\frac{N(p-2)}{2p^2(N+2)}}, & 1 - \frac{(p-2)(N-4)}{2p} < 0 \text{ and } N < p \leq \infty. \end{cases} \end{aligned} \quad (5.28)$$

Proof. Let $w = \mathcal{P}\mathbf{v}^\epsilon - \mathbf{u}$. From (5.2)₃ and (1.4), we have

$$\begin{cases} \partial_t w + w = H, \\ w|_{t=0} = 0, \end{cases} \quad (5.29)$$

where $H = -\mathcal{P}(w \cdot \nabla \mathbf{u}) - \mathcal{P}(\mathbf{v}^\epsilon \cdot \nabla w) - \mathcal{P}(\mathcal{Q}\mathbf{v}^\epsilon \cdot \nabla \mathbf{u}) - \mathcal{P}(\mathbf{v}^\epsilon \cdot \nabla \mathcal{Q}\mathbf{v}^\epsilon)$. Then, by multiplying by w and integrating the resulting equations over \mathbb{R}^N , we have

$$\begin{aligned} & \|w\|_{L^\infty([0,T]; L^2)} + \|w\|_{L^1([0,T]; L^2)} \\ & \leq C \int_0^T (\|w\|_{L^2} (\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{v}^\epsilon\|_{L^\infty}) + \|\mathcal{Q}\mathbf{v}^\epsilon\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2} \\ & \quad + \|\mathbf{v}^\epsilon\|_{L^{\frac{2p}{p-2}}} \|\nabla \mathcal{Q}\mathbf{v}^\epsilon\|_{L^p}) dt, \end{aligned}$$

for all $T > 0$. Using the Gronwall's inequality, we have

$$\begin{aligned} & \|w\|_{L^\infty([0,T]; L^2)} + \|w\|_{L^1([0,T]; L^2)} \\ & \leq C e^{C\|(\nabla \mathbf{u}, \nabla \mathbf{v}^\epsilon)\|_{L_T^1 L^\infty}} (\|\mathcal{Q}\mathbf{v}^\epsilon\|_{L_T^1 L^\infty} \|\nabla \mathbf{u}\|_{L_T^\infty L^2} + \|\mathbf{v}^\epsilon\|_{L_T^\infty L^{\frac{2p}{p-2}}} \|\nabla \mathcal{Q}\mathbf{v}^\epsilon\|_{L_T^1 L^p}), \end{aligned}$$

From Theorem 1.1, we get \mathbf{v}^ϵ uniformly bounded in $L_t^\infty B_{2,1}^\sigma \cap L_t^1 B_{2,1}^\sigma$. From Proposition 5.4, we have

$$\begin{aligned} & \|w\|_{L^\infty([0,T]; L^2)} + \|w\|_{L^1([0,T]; L^2)} \\ & \leq \begin{cases} C\epsilon^{\frac{(p-2)^2 N}{8p(3p-4)}}, & 1 - \frac{(p-2)(N-4)}{2p} > 0 \text{ and } 2 \leq p \leq N, \\ C\epsilon^{\frac{(p-2)^2 N^2}{8p^2(3p-4)}}, & 1 - \frac{(p-2)(N-4)}{2p} > 0 \text{ and } N < p \leq \infty, \\ C\beta\epsilon^{\frac{p-2-\beta}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} = 0, \\ C\epsilon^{\frac{p-2}{4p}}, & 1 - \frac{(p-2)(N-4)}{2p} < 0 \text{ and } 2 \leq p \leq N, \\ C\epsilon^{\frac{N(p-2)}{4p^2}}, & 1 - \frac{(p-2)(N-4)}{2p} < 0 \text{ and } N < p \leq \infty. \end{cases} \end{aligned}$$

By interpolation, we can easily obtain (5.28). \square

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